01

Planar Development of Digital Free-Form Surfaces

Phillip N. Azariadis and Nickolas S. Sapidis
Department of Product & Systems Design Engineering, University of the Aegean, Ermoupolis, Syros, 84100 Greece

Three are the main goals of this chapter. First, a detailed review of differential-geometry criteria for the developability of free-form surfaces is presented. Then, tools measuring accuracy of planar developments are introduced and analyzed. These tools are a prerequisite for evaluating the numerous methods/approaches proposed for generating flat developments of digital free-form surfaces whose level of involvement in many areas of CAD/CAM and Computer Graphics is constantly increasing. Finally, some of the most efficient surface flattening methods are analyzed and categorized followed by discussion of representative examples.

1 Introduction

Planar development of curved surfaces is a well known problem in the manufacturing field. Generating isometric mappings between two different surfaces has also attracted the attention of many researchers in the Computer Graphics community due to its application in non-distorted texture mapping. Historically, this problem first appeared in the context of the "mapmaker" problem: mapping the surface of the Earth onto a plane. Gauss in 1828 proved that such a mapping is not possible due to the different intrinsic curvature of the two surfaces. Thus one can only aim at derivation of an approximately isometric mapping with minimal geometric distortions.

Numerous approaches have been proposed for dealing with the planar development problem of curved surfaces. The problem is trivial when the surface has zero curvature, but becomes significantly complicated when the surface is doubly-curved. In fact, in the latter case, it is proved that there is actually an infinite number of different planar developments for the same surface. Unfortunately, the vast majority of the products produced nowadays include free-form surfaces, which are often created from clouds of points (produced using a modern digitizing device). The selection of the right method for producing a planar development is a subtle task, since there are many issues that have to be taken into consideration, such as production method, material properties, surface geometry, and so on.

The primary purpose of this chapter is to provide an analytical description of developability criteria for surfaces and present tools for measuring the accuracy of planar developments. A classification of the many available methods for generating flat developments of free-form surfaces is also presented. The final goal of this chapter is to help the reader understand, following an intuitive approach, the various aspects of the surface flattening problem and of the related solutions.

This chapter is structured as follows: in Section 2, we review results from differential geometry related to the developability of surfaces. In Section 3, we analyze local properties of a set of affine transformations used to approximate an isometric mapping of a doubly-curved surface onto the plane. In Section 4, a classification of the available flattening methods is presented. Section 5, concludes this chapter with examples and an evaluation of current methods for obtaining adequate planar developments.

2 Criteria of Developability for Surfaces

This section reviews results from differential geometry which are applied to the development of a set of criteria for the *developability* of surfaces. The interested reader is referred to [10,19,21,32] for an in-depth analysis of the presented results. In the following, the cross product of vectors $\mathbf{x}$ and $\mathbf{y}$ is denoted by $\mathbf{x} \times \mathbf{y}$, the gross product of vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ by $[\mathbf{x}, \mathbf{y}, \mathbf{z}]$, and the Euclidean norm of $\mathbf{x}$ by $\|\mathbf{x}\|$.

2.1 Intrinsic Geometry of Surfaces

The surface $\mathbf{x}$ of class C^m, $m \geq 2$, is given by a parametric equation $\mathbf{x} = \mathbf{x}(u, v)$. The differential of $\mathbf{x} = \mathbf{x}(u, v)$ at (u, v) is a 1-1 and onto linear mapping $d\mathbf{x} = \mathbf{x}_\mathbf{u} du + \mathbf{x}_\mathbf{v} dv$ which maps the random vector (du, dv) of the uv-plane to the surface tangent vector $\mathbf{x}_\mathbf{u} du + \mathbf{x}_\mathbf{v} dv$ at $\mathbf{x}(u, v)$. The First Fundamental Form of $\mathbf{x} = \mathbf{x}(u, v)$ is a second degree function of du and dv given as [21]:

$$
\begin{aligned}
\mathrm{I}(du,dv) &= d\mathbf{x}\cdot d\mathbf{x} = \\
&= (\mathbf{x_u}du + \mathbf{x_v}dv)\cdot(\mathbf{x_u}du + \mathbf{x_v}dv) = \\
&= (\mathbf{x_u}\cdot\mathbf{x_u})du^2 + 2(\mathbf{x_u}\cdot\mathbf{x_v})dudv + (\mathbf{x_v}\cdot\mathbf{x_v})dv^2 = \\
&= Edu^2 + 2Fdudv + Gdv^2
\end{aligned} \tag{1}
$$

where

$$E = \mathbf{x_u}\cdot\mathbf{x_u}, \quad F = \mathbf{x_u}\cdot\mathbf{x_v}, \quad G = \mathbf{x_v}\cdot\mathbf{x_v}. \tag{2}$$

The coefficients E, F and G are the *First Order Fundamental Coefficients* and are functions of u and v.

At every point on $\mathbf{x} = \mathbf{x}(u,v)$ there is unit normal vector $\mathbf{N} = \dfrac{\mathbf{x_u}\times\mathbf{x_v}}{\left\|\mathbf{x_u}\times\mathbf{x_v}\right\|}$ which defines an at least C^1 mapping $\mathbf{N}$ from the surface to the unit sphere (Gauss mapping) with respect to u and v. The differential of $\mathbf{N}$ is a vector $d\mathbf{N} = \mathbf{N_u}du + \mathbf{N_v}dv$ normal to the surface tangent plane at $\mathbf{x}(u,v)$ and it is called as the *Second Order Fundamental Form* of surface **x**. It is given by,

$$
\begin{aligned}
\mathrm{II}(du,dv) &= -d\mathbf{x}\cdot d\mathbf{N} = \\
&= -(\mathbf{x_u}du + \mathbf{x_v}dv)\cdot(\mathbf{N_u}du + \mathbf{N_v}dv) = \\
&= Ldu^2 + 2Mdudv + Ndv^2
\end{aligned} \tag{3}
$$

where

$$L = -\mathbf{x_u}\cdot\mathbf{N_u},\ M = -\tfrac{1}{2}(\mathbf{x_u}\cdot\mathbf{N_v} + \mathbf{x}_v\cdot\mathbf{N_u}),\ N = -\mathbf{x_v}\cdot\mathbf{N_v}. \tag{4}$$

The coefficients L, M and N are the *Second Order Fundamental Coefficients* and can be alternatively written as

$$L = \mathbf{x_{uu}}\cdot\mathbf{N}, \quad M = \mathbf{x_{uv}}\cdot\mathbf{N}, \quad N = \mathbf{x_{vv}}\cdot\mathbf{N}. \tag{5}$$

Each surface point $\mathbf{P} = \mathbf{x}(u,v)$ can be characterized through the result of the scalar $\Delta = LN - M^2$ in four distinct cases:

- Elliptic point: $\Delta > 0$
- Hyperbolic point: $\Delta < 0$
- Parabolic point: $\Delta = 0$ and $L^2 + M^2 + N^2 \neq 0$
- Flat point: $\Delta = 0$ and $L = M = N = 0$.

Moreover, there are two scalars, namely κ_1 and κ_2, defined at $\mathbf{P}$ which correspond to the two main curvatures of **x** at $\mathbf{P}$ and are given as the roots of the second degree equation:

$$\kappa^2 - 2H\kappa + K = 0 \tag{6}$$

where

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2) = \frac{EN + GL - 2FM}{2(EG - F^2)} \tag{7}$$

is the *mean curvature* at $\mathbf{P}$ and is the average of κ_1 and κ_2. The scalar

$$K = \kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2} \tag{8}$$

is the well-known *Gaussian curvature* at $\mathbf{P}$. Comparing Eq.(8) with Δ we conclude that a surface point is elliptic, hyperbolic, parabolic or flat iff $K > 0$, $K < 0$ or $K = 0$, respectively.

2.2 Isometric Mappings

Let $\mathbf{f}$ be a 1-1 mapping of a surface S onto a surface S^*. Mapping $\mathbf{f}$ defines an *isometric mapping* or an *isometry* if the arc length of a normal arc $\mathbf{x} = \mathbf{x}(t)$ of S equals the length of its image $\mathbf{x}^* = \mathbf{x}^*(t) = \mathbf{f}(\mathbf{x}(t))$ on S^*. If $\mathbf{f}$ is an isometry then $\mathbf{f}^{-1}$ is an isometry from S^* onto S, too. In such a case we call S and S^* isometric. In general, it can be proved [21] that a 1-1 mapping $\mathbf{f}$ from S onto S^* is an isometry iff for each part $\mathbf{x} = \mathbf{x}(u, v)$ of S and $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u, v))$ of S^* the first order fundamental coefficients are equal:

$$E = E^*, \; F = F^* \text{ and } G = G^*. \tag{9}$$

2.3 Developability Criteria for Surfaces

Two surfaces S and S^* are called *applicable* if there is a continuous family of mappings $\mathbf{f}_\lambda$, $0 \le \lambda \le 1$, from S onto S^* such that

a. $\mathbf{f}_\mathbf{0}(S) = S$
b. $\mathbf{f}_\mathbf{1}(S) = S^*$
c. mappings $\mathbf{f}_\lambda$ are isometric mappings from S onto $\mathbf{f}_\lambda(S)$, for every $\lambda \in [0,1]$.

Intuitively, one understand that S and S^* are applicable if S can be bended continuously and isometrically in such a way that the final bended surface coincides with S^*. Obviously, if S and S^* are applicable then they are isometric too.

Taking the aforementioned into account, we can easily ascertain that a surface is *developable* iff it is applicable to the plane. Thus, every developable surface can be unrolled onto a plane without distortion, implying that a developable surface can be constructed through the smooth bend of a plane sheet. As a result, every developable surface is isometric to the plane and therefore it holds that $LN - M^2 = 0$ – meaning that all points of such a surface are parabolic or flat – and thus its Gaussian curvature equals zero at every point. This discussion is summarized by the following theorem:

Theorem 1. *Let S be a C^m, $m \geq 2$, class surface. The following properties are equivalent:*

i. *The surface is developable.*
ii. *All the surface points are parabolic or flat.*
iii. *The Gaussian curvature at every surface point equals zero.*
iv. *The surface is applicable to the plane.* ■

Essentially, this theorem stands as a general criterion for the developability of any surface. However, more specific criteria can be developed for certain families of surfaces like ruled or revolved surfaces.

2.3.1. Ruled Surfaces

A ruled surface is constructed from a one-parameter family of lines and its normal parametric equation is given by

$$\mathbf{x} = \mathbf{x}(u,v) = \mathbf{n}(u) + v\mathbf{q}(u), \; v \in (-\infty, +\infty), \tag{10}$$

where, $\mathbf{n} = \mathbf{n}(u)$ is a C^m, $m \geq 2$, curve called a *directrix* and $\mathbf{q} = \mathbf{q}(u)$ is a C^m, $m \geq 2$, vector function which defines the lines' direction at each point $\mathbf{n} = \mathbf{n}(u)$, and it is called a *generatrix*. The surface normal vector is given by,

$$\mathbf{N} = \frac{(\dot{\mathbf{n}} + v\dot{\mathbf{q}}) \times \mathbf{q}}{\sqrt{(\dot{\mathbf{n}} + v\dot{\mathbf{q}})^2 - (\mathbf{n},\mathbf{q})^2}} \tag{11}$$

where, $\dot{\mathbf{n}} = \dfrac{d\mathbf{n}}{du}$ and $\dot{\mathbf{q}} = \dfrac{d\mathbf{q}}{du}$.

Theorem 2. *A ruled surface is developable iff* $[\dot{\mathbf{n}}, \mathbf{q}, \dot{\mathbf{q}}] = 0$.

Proof. A ruled surface is developable if the cross product of two normal vectors defined at arbitrarily positions $v_1 \neq v_2$ of the same generatrix equals to zero. It holds,

$$\begin{gathered} \left[(\dot{\mathbf{n}} + v_1\dot{\mathbf{q}}) \times \mathbf{q}\right] \times \left[(\dot{\mathbf{n}} + v_2\dot{\mathbf{q}}) \times \mathbf{q}\right] = \\ = [(\dot{\mathbf{n}} + v_1\dot{\mathbf{q}}), \mathbf{q}, \mathbf{q}](\dot{\mathbf{n}} + v_2\dot{\mathbf{q}}) - [(\dot{\mathbf{n}} + v_1\dot{\mathbf{q}}), \mathbf{q}, (\dot{\mathbf{n}} + v_2\dot{\mathbf{q}})]\mathbf{q} = \\ = (v_1 - v_2)[\dot{\mathbf{n}}, \mathbf{q}, \dot{\mathbf{q}}]\mathbf{q} \end{gathered} \tag{12}$$

which completes the proof. ■

2.3.2. Revolved Surfaces

Let the normal parametric equation of a revolved surface be

$$\mathbf{x} = \mathbf{x}(u,v) = \left(r(u)\cos v, \quad r(u)\sin v, \quad z(u)\right) \tag{13}$$

where $r = r(u)$, $z = z(u)$ are the parametric equations of the surface meridians, and u is the arc length of each meridian.

Theorem 3. *A revolved surface is developable iff* $r'' = 0$.

Proof. The first order fundamental coefficients are

$$E = 1\,,\ F = 0\,,\ G = r^2 \tag{14}$$

$$L = r'z'' - r''z'\,,\ M = 0\,,\ N = rz' \tag{15}$$

Substituting into Eq.(8) leads to the following expression for the Gaussian curvature of the surface

$$K = K(u) = \frac{z'(r'z'' - r''z')}{r}\,, \tag{16}$$

which obviously depends only on the arc length u. Then,

$$(r')^2 + (z')^2 = 1 \Rightarrow r'r'' + z'z'' = 0 \Rightarrow z'' = -\frac{r'r''}{z'}\,. \tag{17}$$

Substituting Eq.(17) into Eq. (16) produces

$$K(u) = -\frac{r''}{r}\,, \tag{18}$$

which completes the proof. ■

Theorem 4. *A revolved surface is developable iff* $\begin{vmatrix} r'(u_1) & z'(u_1) \\ r'(u_2) & z'(u_2) \end{vmatrix} = 0$ *for every* $u_1 \neq u_2$.

Prof. The surface normal vector is

$$\mathbf{N} = \left(-z'(u)\cos v, -z'(u)\sin v, r'(u)\right). \tag{19}$$

Let $\mathbf{N_1}$ and $\mathbf{N}_2$ two surface normal vectors defined at the same meridian for $u_1 \neq u_2$. The surface will be developable iff the cross product of $\mathbf{N_1}$ and $\mathbf{N}_2$ equals to zero. This implies that

$$\mathbf{N_1} \times \mathbf{N_2} = \begin{vmatrix} r'(u_1) & z'(u_1) \\ r'(u_2) & z'(u_2) \end{vmatrix} (\sin v, -\cos v, 0)\,. \tag{20}$$

Taking into account that $\sin^2 v + \cos^2 v = 1$, we derive that $\mathbf{N_1} \times \mathbf{N_2} = \mathbf{0}$ iff $\begin{vmatrix} r'(u_1) & z'(u_1) \\ r'(u_2) & z'(u_2) \end{vmatrix} = 0$, which completes the proof. ■

3 Evaluating Planar Developments

The above criteria for developability limit the variety of surfaces that can be isometrically unfolded onto the plane. On the other hand, the vast variety of surfaces used in today's products are doubly curved with arbitrarily complex shapes. For these surfaces, approximate, local isometric-mappings should be considered, which is the subject of this section.

The surfaces considered in the present context are approximated with an adequate mesh $\mathbf{\Phi}$ of triangles. This implies that the mesh is allowed to have variable density depending on the local accuracy of approximation. Note, that $\mathbf{\Phi}$ contains only positive non-degenerated triangles, i.e., the vertices have a counter-clockwise order and the triangle area is always positive.

3.1 Affine Triangle Transformations

Let S be a surface given by the parametric equation $\mathbf{x} = \mathbf{x}(u, v)$ and the uv-plane P. Then, $\mathbf{x}^{-1}$ maps three-dimensional points of S onto the plane P. If $\mathbf{x}$ is an isometry then the surface S is developable. We focus on the case where $\mathbf{x}$ is not an isometry.

Since S is approximated with a finite number of triangular elements, we can also approximate $\mathbf{x}$ using the same number of local mappings between triangular elements. We assume that there is a mesh $\boldsymbol{\varphi}$ on plane P having equivalent topological characteristics with $\boldsymbol{\Phi}$. There is a 1-1 correspondence between the elements of $\boldsymbol{\Phi}$ and $\boldsymbol{\varphi}$.

Let us consider an arbitrarily pair of corresponding triangles $\nabla(\mathbf{A}, \mathbf{B}, \mathbf{C})$ of $\boldsymbol{\Phi}$ and $\nabla(\mathbf{a}, \mathbf{b}, \mathbf{c})$ of $\boldsymbol{\varphi}$, where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \Re^3$ and $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \Re^2$ are respectively the vertices of the triangles. Using the Gram-Schmidt orthogonalization, we define a local orthonormal coordinate system in each triangle as follows.

Let L be the orthonormal coordinate system with unit vectors $\mathbf{Q}_1$ and $\mathbf{Q}_2$ defined as

$$\begin{aligned} \mathbf{Q}_1 &= \frac{\mathbf{B} - \mathbf{A}}{\|\mathbf{B} - \mathbf{A}\|} \\ \mathbf{Q}_2 &= \frac{(\mathbf{C} - \mathbf{A}) - ((\mathbf{C} - \mathbf{A}) \cdot \mathbf{Q}_1)\mathbf{Q}_1}{\|(\mathbf{C} - \mathbf{A}) - ((\mathbf{C} - \mathbf{A}) \cdot \mathbf{Q}_1)\mathbf{Q}_1\|} \end{aligned} \tag{21}$$

The origin of L is set to be the point $\mathbf{A}$.
Similarly, let ℓ denote the local orthonormal coordinate system with unit vectors $\mathbf{q}_1$ and $\mathbf{q}_2$:

$$\begin{aligned} \mathbf{q}_1 &= \frac{\mathbf{b} - \mathbf{a}}{\|\mathbf{b} - \mathbf{a}\|} \\ \mathbf{q}_2 &= \frac{(\mathbf{c} - \mathbf{a}) - ((\mathbf{c} - \mathbf{a}) \cdot \mathbf{q}_1)\mathbf{q}_1}{\|(\mathbf{c} - \mathbf{a}) - ((\mathbf{c} - \mathbf{a}) \cdot \mathbf{q}_1)\mathbf{q}_1\|} \end{aligned} \tag{22}$$

The origin of ℓ is the point $\mathbf{a}$. Then, the local coordinates of the two triangles are given, with respect to the two local coordinate systems, by

$$\begin{aligned} \mathbf{A}^L &= (0, 0) \\ \mathbf{B}^L &= \|\mathbf{B} - \mathbf{A}\|\mathbf{Q}_1 = (B_x^L, 0) \\ \mathbf{C}^L &= ((\mathbf{C} - \mathbf{A}) \cdot \mathbf{Q}_1, (\mathbf{C} - \mathbf{A}) \cdot \mathbf{Q}_2) = (C_x^L, C_y^L) \end{aligned} \tag{23}$$

and

$$
\begin{aligned}
\mathbf{a}^{\ell} &= (0,0) \\
\mathbf{b}^{\ell} &= \|\mathbf{b}-\mathbf{a}\|\mathbf{q}_1 = \left(b_x^{\ell},0\right) \\
\mathbf{c}^{\ell} &= \left((\mathbf{c}-\mathbf{a})\cdot\mathbf{q}_1,(\mathbf{c}-\mathbf{a})\cdot\mathbf{q}_2\right) = \left(c_x^{\ell},c_y^{\ell}\right)
\end{aligned}
\tag{24}
$$

We define an affine transformation of triangle $\nabla(\mathbf{A},\mathbf{B},\mathbf{C})$ to the triangle $\nabla(\mathbf{a},\mathbf{b},\mathbf{c})$ by a local linear mapping **f** written in matrix form as

$$
\begin{bmatrix} b_x^{\ell} & c_x^{\ell} \\ 0 & c_y^{\ell} \end{bmatrix} = \mathbf{f}\begin{bmatrix} B_x^{\ell} & C_x^{\ell} \\ 0 & C_y^{\ell} \end{bmatrix} \Leftrightarrow
$$

$$
\mathbf{f} = \begin{bmatrix} b_x^{\ell} & c_x^{\ell} \\ 0 & c_y^{\ell} \end{bmatrix}\begin{bmatrix} B_x^{\ell} & C_x^{\ell} \\ 0 & C_y^{\ell} \end{bmatrix}^{-1} \Leftrightarrow
$$

$$
\mathbf{f} = \begin{bmatrix} \dfrac{b_x^{\ell}}{B_x^{L}} & \dfrac{B_x^{L}c_x^{\ell} - b_x^{\ell}C_x^{L}}{B_x^{L}C_y^{L}} \\ 0 & \dfrac{c_y^{\ell}}{C_y^{L}} \end{bmatrix} \tag{25}
$$

Note that the matrix $\begin{bmatrix} B_x^{\ell} & C_x^{\ell} \\ 0 & C_y^{\ell} \end{bmatrix}$ always has an inverse since the determinant $D = B_x^{\ell}C_y^{\ell}$ is nonzero. The set of all local mappings **f** defines an approximation of $\mathbf{x}^{-1}$. Furthermore, if the two corresponding triangles are equal then **f** is an isometry and surface S is considered, in this triangular area, locally isometric to the plane. Thus, we should focus on the study of the properties of **f**, which is the subject of the next section.

3.2 Properties of Local Mappings

In this section we shall investigate both quantitative and qualitative characteristics of local mappings **f** in order to ascertain whether they define an isometry or not.

3.2.1 Mapping Points

We can simplify the used notation taking into account that, since L and ℓ are orthonormal coordinate systems, we can express the coordinates of the vertices of $\nabla\left(\mathbf{A}^{L},\mathbf{B}^{L},\mathbf{C}^{L}\right)$ and $\nabla\left(\mathbf{a}^{\ell},\mathbf{b}^{\ell},\mathbf{c}^{\ell}\right)$ in a global coordinate system W as

$$
\begin{array}{lll}
\mathbf{A}^W = (0,0) & & \mathbf{a}^W = (0,0) \\
\mathbf{B}^W = (B_x^L, 0) & \text{and} & \mathbf{b}^W = (b_x^\ell, 0) \\
\mathbf{C}^W = (C_x^L, C_y^L) & & \mathbf{c}^W = (c_x^\ell, c_y^\ell)
\end{array}
\tag{26}
$$

In addition, we drop the use of superscripts and, in the rest of this section, we denote the triangle vertices of Eq.(26) as (**A**,**B**,**C**) and (**a**,**b**,**c**) respectively. This is illustrated in Fig.1.

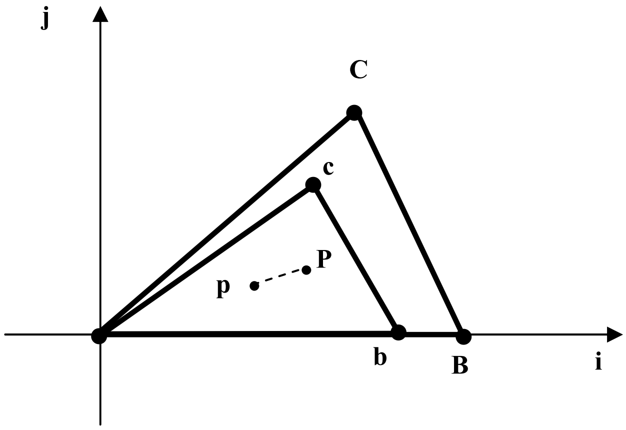

Fig. 1. *The two corresponding triangles drawn in the global coordinate system W with unit vectors* ***i****=(1,0) and* ***j****=(0,1)*

Let **P**=(X,Y) a point within triangle $\nabla(\mathbf{A},\mathbf{B},\mathbf{C})$. This point is mapped through **f** onto a point **p**=(x,y) within triangle $\nabla(\mathbf{a},\mathbf{b},\mathbf{c})$ according to $\mathbf{p} = \mathbf{fP}$. If **f** is an isometry then the squared Euclidean-distance d^2 between **p** and **P** should be zero. This distance is computed as follows

$$
\begin{aligned}
&\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{f} \begin{bmatrix} X \\ Y \end{bmatrix} \Leftrightarrow \\
&\begin{bmatrix} X \\ Y \end{bmatrix} = \mathbf{f}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow \begin{bmatrix} X \\ Y \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{f}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow \\
&\begin{bmatrix} X - x \\ Y - y \end{bmatrix} = (\mathbf{f}^{-1} - \mathbf{I}) \begin{bmatrix} x \\ y \end{bmatrix}
\end{aligned}
\tag{27}
$$

Then

$$
\begin{aligned}
d^2 &= \begin{bmatrix} X - x & Y - y \end{bmatrix} \begin{bmatrix} X - x \\ Y - y \end{bmatrix} = \\
&= \begin{bmatrix} x & y \end{bmatrix} (\mathbf{f}^{-1} - \mathbf{I})^{\mathrm{T}} (\mathbf{f}^{-1} - \mathbf{I}) \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow
\end{aligned}
$$

$$d^2 = \mathbf{p}^{\mathrm{T}}\mathbf{M}\mathbf{p}\,, \tag{28}$$

where $\mathbf{M} = (\mathbf{f}^{-1} - \mathbf{I})^{\mathrm{T}}(\mathbf{f}^{-1} - \mathbf{I})$. The scalar d^2 expresses quantitatively the distortion caused by **f** regarding the mapping of **P** onto **p**. Furthermore, expanding Eq.(28) we find that the mapped point **p** belongs to an ellipse which can be computed according to the following theorem.

Theorem 5. *Any point **P**=(X,Y) is mapped through **f** into a point **p**=(x,y) which belongs to the ellipse*

$$(m_{11} - 1)x^2 + \left(m_{12} + m_{21}\right)xy + (m_{22} - 1)y^2 + 2Xx + 2Yy - X^2 - Y^2 = 0$$

where $\mathbf{M} = [m_{ij}] = (\mathbf{f}^{-1} - \mathbf{I})^{\mathrm{T}}(\mathbf{f}^{-1} - \mathbf{I})$, $i, j = 1,2$ ■

Now, we focus on estimating the error introduced by the mapping **f**. For this purpose, we apply **f** on a circle of unit radius centered at (0,0). This transformation will reveal more detailed characteristics of the distortion that **f** may cause if it is not an isometry.

Let us consider a circle $\mathbf{X_c}(\omega) = \left(X_c(\omega), Y_c(\omega)\right) = \left(\cos\omega, \sin\omega\right)$ of unit radius with its center lying at the origin of the global system of reference. An arbitrarily circle point is mapped, through $\mathbf{f} = \left[f_{ij}\right]$, to a point $\mathbf{x}_c\left(\omega\right)$ which belongs to the ellipse,

$$\begin{bmatrix} x_c(\omega) \\ y_c(\omega) \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ 0 & f_{22} \end{bmatrix} \begin{bmatrix} X_c(\omega) \\ Y_c(\omega) \end{bmatrix} \Rightarrow$$

$$\begin{aligned} x_c(\omega) &= f_{11}\cos\omega + f_{12}\sin\omega \\ y_c(\omega) &= f_{22}\sin\omega \end{aligned} \tag{29}$$

Then, the squared distance between the new point $\left(x_c(\omega), y_c(\omega)\right)$ and the origin is

$$\begin{aligned} g(\omega) &= x_c^2(\omega) + y_c^2(\omega) = \\ &= f_{11}^2\cos^2\omega + (f_{12}^2 + f_{22}^2)\sin^2\omega + 2f_{11}f_{12}\cos\omega\sin\omega \end{aligned} \tag{30}$$

The angle at which the unit circle suffers the maximum or minimum deformation is computed at $g'(\omega) = 0$ as

$$\omega_1 = \frac{1}{2}\tan^{-1}\frac{2f_{11}f_{12}}{f_{11}^2 - f_{12}^2 - f_{22}^2} \tag{31}$$

Substituting Eq.(31) into Eq.(30) the principal direction of the ellipse relative to the x-axis of the global system is given by

$$\varphi = \tan^{-1}\left(y_c(\omega_1) \big/ x_c(\omega_1) \right) \tag{32}$$

The extension or shrinkage d_p of the unit circle along this direction is

$$d_p = \sqrt{x_c^2(\omega_1) + y_c^2(\omega_1)}\,. \tag{33}$$

In a similar fashion the second component of the extension is

$$d_q = \sqrt{x_c^2\left(\omega_1 + \pi/2\right) + y_c^2\left(\omega_1 + \pi/2\right)} \tag{34}$$

Having computed the lengths and the directions of the ellipse's axis, we can write the parametric equation of the ellipse as,

$$\left.\begin{aligned} x_e(u) &= d_p \cos\varphi \cos u - d_q \sin\varphi \sin u \\ y_e(u) &= d_p \sin\varphi \cos u + d_q \cos\varphi \sin u \end{aligned}\right\}, u \in [0, 2\pi] \tag{35}$$

Based on the above analysis we can state the following theorem [2].

Theorem 6. *A circle of unit radius is transformed through **f** into an ellipse with half axis lengths equal to* d_p *and* d_q*, respectively, and with its major axis inclined with an angle φ relative to the x-axis of the global system of reference.* ■

Remark 1. *The ellipse components* d_p *and* d_q *can be also computed through the Singular Value Decomposition of **f**.*

In other words, **f** can be expressed as a matrix product $\mathbf{f} = \mathbf{R}(\theta)\boldsymbol{\Lambda}\mathbf{R}(\varphi)$, where $\mathbf{R}(\varphi)$ and $\mathbf{R}(\theta)$ express rotations while $\boldsymbol{\Lambda}$ is a diagonal matrix expressing the deformation along the two principal axis of the ellipse, i.e., $\boldsymbol{\Lambda} = \begin{bmatrix} d_p & 0 \\ 0 & d_q \end{bmatrix}$. Thus, d_p and d_q are the square roots of the eigenvalues of the positive matrix $\mathbf{ff^T}$.

Remark 2. *The mapping **f** is an isometry iff* $d_p = 1$ *and* $d_q = 1$.

Remark 3. *The determinant of the Jacobian of mapping **f** equals* $d_p d_q$.

Obviously, a trivial case is when $\mathbf{f} = \mathbf{I_2}$.

3.2.2 Measuring the Accuracy of Planar Developments

Taking into account the set of mappings **f** which approximate $\mathbf{x}^{-1}$, we wish to derive meaningful indices measuring the metric distortion during the planar development of a doubly-curved surface.

Homogeneity of distortion: The distortion should be homogeneous throughout the surface in order to avoid rapid changes in local areas of the planar development. The ratio of the minimum over the maximum value of d_p and the corresponding ratio for d_q are good measures of the distortion variation in the first and second principal directions. The ratio of $\min\{d_p d_q\}$ over $\max\{d_p d_q\}$, for all triangles, characterizes the homogeneity of the distortion along both principal directions, i.e.,

$$h = \frac{\min\{d_p d_q\}}{\max\{d_p d_q\}}. \tag{36}$$

Ideally, the value of h should be constant over the surface and close to the unit.

Aspect ratio: Aspect ratio should be preserved to avoid non-uniform stretching of the planar development. This distortion can be expressed as

$$r = \frac{\min\{d_p\}}{\max\{d_q\}} \tag{37}$$

for all the elements of $\mathbf{\Phi}$.

These indices may be used to measure the accuracy of a planar development of a doubly-curved surface both locally and globally. Such applications will be given later in this chapter.

4 Methods for Approximate Planar Development of Curved Surfaces

The problem of flattening a curved surface onto a plane is important not only for manufacturing but also for the computer graphics community. In fact, the well known two-dimensional texture mapping technique is fundamentally equivalent to the planar development of three-dimensional surfaces. Thus, many attempts have been made to solve the problem of producing nearly isometric flat developments of curved surfaces both for manufacturing and texture mapping purposes. Most of these methods may be classified into three categories:

A. Methods based on the minimization of an objective function (usually called as *energy function*), assuming a degree of elasticity in the surface material.

B. Methods based on intrinsic differential-geometric properties of surfaces. These are usually employed when local accuracy in the planar development is of great importance, or when it is necessary to insert line cuttings.

C. Methods based on the approximation of the initial surface with developable surfaces which are isometrically unfolded onto the plane.

Current methods representing all categories are presented in the following subsections.

4.1 Category A: Minimization of an Energy Function

These methods employ objective functions measuring the ‘‘difference’’ between the planar development and the corresponding three-dimensional surface. In other words, these functions measure the energy needed for the planar development to be fitted onto the free-form surface or vice-versa. The closer to zero the objective-function's value is, the better planar development is obtained. Clearly, only for a developable surface the corresponding objective function may obtain a zero value. Minimization of the objective function is usually achieved using a standard optimization method, e.g., an iterative technique.

Ma and Lin [30] were the first to present a flattening technique based on optimizing an objective function comparing the length of triangle edges of the surface mesh $\mathbf{\Phi}$ with that of corresponding triangle edges in the planar mesh $\boldsymbol{\varphi}$. Unfortunately, this method may produce triangles on the plane *P* with the wrong orientation, leading to a planar development with overlaps (see Fig. 2).

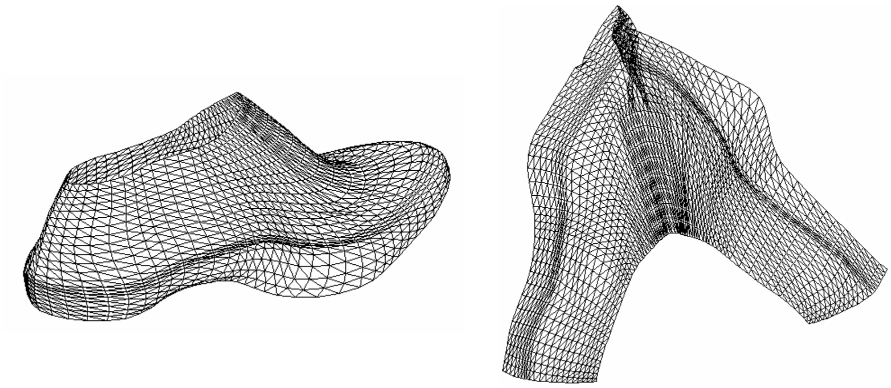

Fig. 2. *Example of applying the method of Ma and Lin to derive a planar development of a shoe last.*

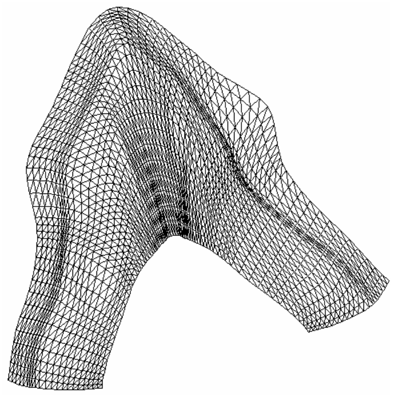

Fig. 3. *Using the method of Maillot et al., it is possible to derive a planar development of the shoe last of Fig. 2 without overlapping areas.*

Maillot et al. [17] improve the previous method by using a new objective function, linearly combining an energy function comparing lengths with another one based on the difference of signed areas, which avoids definition of triangles with the wrong orientation. This is evident in the example of Fig. 3. An important disadvantage of both methods is that, in order for them to converge, one must produce a good initial estimate of the planar development. Ma & Lin propose no solution for this problem, while Maillot et al. offer a technique not applicable to all surfaces.

Azariadis and Aspragathos [2] further improve the aforementioned method by modifying the area energy function and by giving a solution to the initial guess problem. They also introduce an algorithm for preserving, during the flattening process, either isoparametric curves [2] or arbitrary curves [3]. The usefulness of this property has been verified by using industrial examples. In [3], it is experimentally shown that if the mesh $\mathbf{\Phi}$ approximates the curved surface with a sufficient accuracy then further refinement of $\mathbf{\Phi}$ has almost no effect on the final result. However, estimation of the minimum size of the mesh $\mathbf{\Phi}$**,** sufficient for accurate planar-development, remains an open problem.

Employing the material properties of the initial curved-surface, Shimada & Tada [28,29] proceed to developing a method based on the theory of finite-elements to construct planar developments of arbitrary three-dimensional surfaces. More specifically, these researchers propose an approximation based on solving a planar stress-problem using triangular elements. The related objective function is minimized using a particular iterative process instead of a classical optimization-algorithm. An important advantage of this method is that it does not usually require an accurate initial estimate of the solution to converge. However, the method may fail if the geometry of the surface is sufficiently complex or if overlaps appear in the planar development.

Another method is proposed by Bennis et al. [7], where a relaxation procedure is used for the homogeneous distribution of deformation of the geodesic-curvature error in the planar development. A limitation of this technique is its strong dependence on the surface parameterization and its initialization by specification of a surface parametric-curve which is mapped onto the plane. Regarding trimmed surfaces, this method often requires decomposing the initial surface into subparts of very simple geometry.

Yu et al.[14] present an algorithm for optimal development of a smooth continuous curved surface onto the plane. The development process is modeled using in-plane strain from the curved surface to its planar development. Minimization of strain in the planar development is achieved by solving a constrained nonlinear programming problem. Another approach [8, 9] formulates the planar development problem using a spring-mass system and calculating the strain energy released during flattening. These authors also use a color graph to indicate areas where cutting lines should be introduced to release more strain energy.

Sheffer and de Sturler [27] introduce a method based on the observation that a triangulated planar mesh is fully defined by the mesh angles up to global scaling, rotation, and translation. The authors formulate the parameterization problem in terms of the flat-mesh angles and solve it in the angle space. The method also involves constraints on angles defining a valid (continuous) planar mesh. The main part of the method is minimization of the angular distortion of the parameterization, subject to the above constraints. Recently, the authors enhanced their flattening technique by minimizing both angular distortion and linear distortion [27]. The authors claim that this revised method avoids foldovers in the derived planar development.

4.2 Category B: Employing Intrinsic Differential-Geometric Properties of Surfaces

This category includes flattening methods that use intrinsic differential-geometric properties of surfaces like the Gaussian curvature or the geodesic curvature. Taking under consideration the characteristics of geodesic curves of a surface, Manning[22] develops a flattening method based also on an ‘‘isometric tree’’, i.e., a network of surface points connected to each other with edges. Eventually, this tree is projected onto a plane using an isometric mapping.

A method based on properties of the Gaussian curvature of a surface is proposed by Hinds et al.[15] aiming at planar developments for apparel design. More specifically, since clothing manufacturing requires that planar developments are free of foldovers, the authors focus on developing a flattening technique that fulfils this requirement and thus, almost always, produces developments with openings, called ‘‘radial developments’’. McCartney et al.[23] offer another method, aiming again at the clothing industry, which handles the insertion of darts and gussets by creating appropriate openings.

Parida and Mudur [24] deal with the special case of composite materials and propose a robust flattening technique based on constraints. Azariadis and Aspragathos [1] extend this method to a general-purpose surface flattening technique divided into three-stages. In the first stage, an adequate guide-strip is located on the triangulated surface. Using this guide-strip an initial planar development is derived by isometrically unfolding triangle strips onto the plane. At the final stage, foldovers and cuts are eliminated according to certain criteria. A more elaborated approach to the planar development refinement is introduced in [4] where a special genetic algorithm has been developed for global optimization under constrains.

Wolfson and Schwartz [12], and Schwartz, Shaw and Wolfson [11] used a special MDS (Multi-Dimensional Scaling) approach to flatten the curved surface using geodesic distances, and by minimizing the functional presented by Sammon in [16], which resembles the Stress-1 functional. Their method involves high computational complexity and therefore is not practical. Zigelman el al.[33] improved this method by introducing a new mapping method that preserves both the local and the global structure of the planar development, with minimal shearing effects.

4.3 Category C: Approximation with Developable Surfaces

These methods subdivide the initial three-dimensional surface into pieces, which are approximated with developable patches. These patches are defined by a one-parameter envelope of tangent planes, which intersect pairwise and define a line in the three-dimensional space. Thus, each plane is tangent to the constructed developable patch along this straight line, which is called ‘‘generatrix’’. Subdivision of the initial surface into such patches is performed so that the surface produced is at least C^0 continuous.

A method for the construction of developable surfaces, along the lines of the above methodology, is proposed by Redont [25]. The user derives developable surfaces by specifying the orientation of the tangent plane along a geodesic. Clearly, this is not a practical method as it is very hard for any user to define appropriate orientation of the tangent planes so that the desirable developable surface is constructed.

Bodduluri and Ravani [5,6] develop a method for the design of developable surfaces based on the concept of duality between points and planes in the three-dimensional projective space, giving a new representation for developable surfaces in the context of ‘‘plane geometry’’. The developable surface is designed using control planes (which are dual to points). Fitting is performed employing existing techniques for curve design, like Bezier or B-spline fitting.

A simple method for the approximation of an arbitrary curved surface with developable patches is proposed in [13]. More specifically, the author adopts the approach that a developable surface may be represented as an appropriate ruled-surface. On the basis of a developability condition for ruled surfaces, a simple algorithm is proposed for subdividing a surface into a set of developable ruled-surfaces, within a given approximation error. The result of this algorithm is a C^0 composite-surface consisting of developable surfaces.

Many techniques have appeared which face the problem of designing developable surfaces under specific conditions related to the nature of a particular problem. For example, Sundar and Varada [31] propose a method for calculating developments of ducts, whose surface is approximated with developable surfaces. Also, Hoschek [18] proposes a method for deriving approximately-developable surfaces from surfaces of revolution. Finally, Leopoldseder and Pottmann [20] present a method for designing/representing approximately-developable surfaces using conic segments. Although they deal with the problem using a new approach, based on the duality between points and planes in the Euclidean space, their method may be considered as an extension of that proposed by Redont [25].

5 Application and Evaluation of Current Methods

We conclude this chapter by presenting a series of examples of planar developments of doubly-curved surfaces. First, let as briefly describe one of the most commonly used approaches for surface flattening based on an energy model.

5.1 The modified length-area energy model

One of the most commonly used energy models for deriving planar developments of doubly-curved surfaces is the one proposed by Ma[30] and latter extended by Mailot [17]. This energy function is actually a convex combination of two energy functionals expressing the metric distortion in terms of length and signed area

$$E(\mathbf{x}) = aE_{length}(\mathbf{x}) + (1-a)E_{area}(\mathbf{x}) \tag{38}$$

where $0 \le a \le 1$. The length functional is

$$E_{length} = 2\sum_{\mathbf{M_i}\in\mathbf{\Phi}} \sum_{\mathbf{M_j}\in\Omega_i} \frac{\left(\left\|\mathbf{m_i}-\mathbf{m_j}\right\|^2 - \left\|\mathbf{M_i}-\mathbf{M_j}\right\|^2\right)^2}{\left\|\mathbf{M_i}-\mathbf{M_j}\right\|^2} \tag{39}$$

where $\mathbf{M_i}$ and $\mathbf{m_i}$ are vertices of triangles of $\mathbf{\Phi}$ and $\boldsymbol{\varphi}$, respectively. Ω_i is the set of vertices of all triangle edges of $\mathbf{\Phi}$ meeting at $\mathbf{M_i}$ ($i \ne j$). The signed-area functional is

$$E_{area} = \sum_{M_i\in\mathbf{\Phi}} \sum_{(j,k)\in V_i} \frac{\left(\det(\mathbf{m_i m_j}, \mathbf{m_i m_k}) - \left\|\mathbf{M_i M_j} \times \mathbf{M_i M_k}\right\|\right)^2}{\left\|\mathbf{M_i M_j} \times \mathbf{M_i M_k}\right\|} \tag{40}$$

where $V_i = \left\{\text{all pairs } (j,k) : (\mathbf{M_i}, \mathbf{M_j}, \mathbf{M_k}) \text{ define a triangle in } \mathbf{\Phi}\right\}$ and $i \ne j \ne k$. Eq.(40) has been proposed in [3] in order to express the area functional with respect to mesh vertices instead of mesh triangles as it was in [17]. This modification simplifies gradient computations. The value of α is determined with respect to the complexity of the surface geometry. Usually a value around 0.5 produces acceptable results with no foldovers. However, for surfaces with high curvature areas smaller values are required.

5.2 Examples

In this section we compare the modified length-area energy (LAE) algorithm [3] with the FEA algorithm [28,29] using as first benchmark the development of the landscape shown in Fig.4, a significantly complex surface with high-curvature areas. This example is quite representative for texture mapping applications.

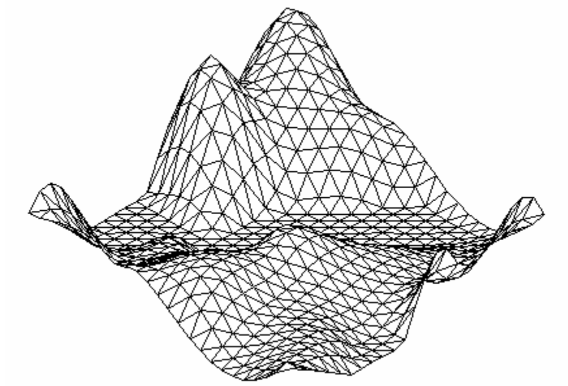

Fig. 4. *A landscape surface with high-curvature areas.*

Due to the complex geometry we have to tune the parameter α of Eq.(38) in order to avoid foldovers in the final planar development. A proper value for α is found to be 0.2 (see Fig.5). On the other hand, the FEA method requires no tuning; it directly produces a planar development without overlapping regions (Fig.6). Both planar developments are filled with colors indicating the areas with low/high distortion. The darker the color, the higher the distortion is in the planar development. The distortion is measured by comparing the edge lengths of triangles in the surface to those in the planar development.

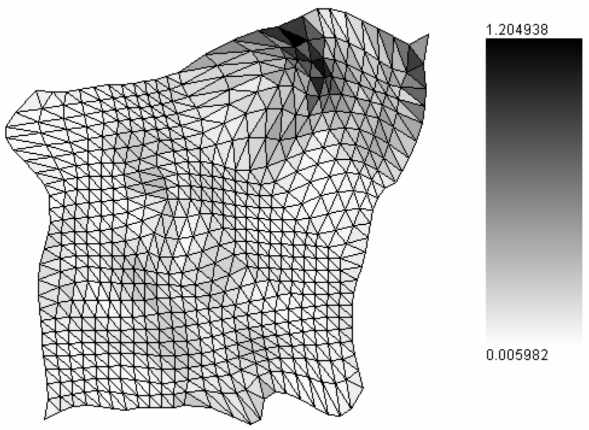

Fig. 5. *The planar development derived using the modified LAE model.*

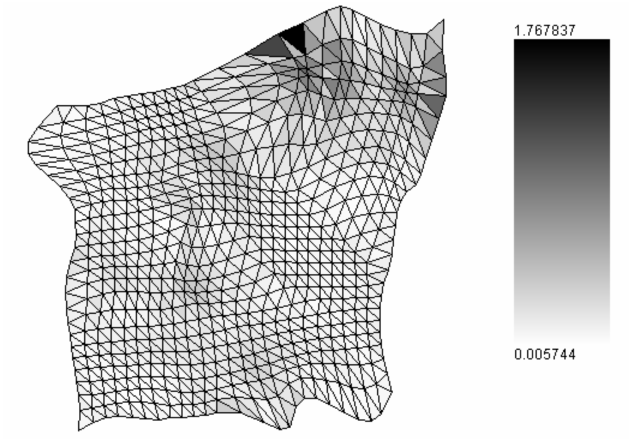

Fig. 6. *The planar development derived using the FEA flattening method.*

Table 1. *Landscape and last examples: Analysis of planar developments.*

Flattening Method / Surface	Homogeneity index along 1st direction $\frac{\min d_p}{\max d_p}$	Homogeneity index along 2nd direction $\frac{\min d_q}{\max d_q}$	Homogeneity index $\frac{\min d_p d_q}{\max d_p d_q}$	Aspect ratio $\frac{\min d_p}{\max d_q}$	Elapsed time (sec)
LAE / Landscape	0.216880	0.266572	0.344655	0.578145	3.365
FEA/ Landscape	0.253818	0.209432	0.183896	0.485494	106.743
LAE / Last	0.687047	0.628918	0.708005	0.871228	64.894
FEA / Last	0.667407	0.698568	0.741347	0.670895	6.209

Comparing the two planar developments reveals some important features: It is obvious that areas with higher distortion correspond to areas with higher curvature, as it was expected. Also, the planar development derived with the LAE model is significantly less distorted (around 1.47 times) than that obtained using FEA, which is apparent in the two color maps. An in depth analysis of the accuracy/quality of both planar developments, with respect to the indices introduced in this chapter, is presented in Table 1. The data in this table establish that the development based on the LAE model is better than that produced using the FEA method: The homogeneity index (Eq.(36)) for LAE is almost twice that of FEA. This is very important for applications like texture mapping where sudden changes in the image quality are easily noticed by the human eye.

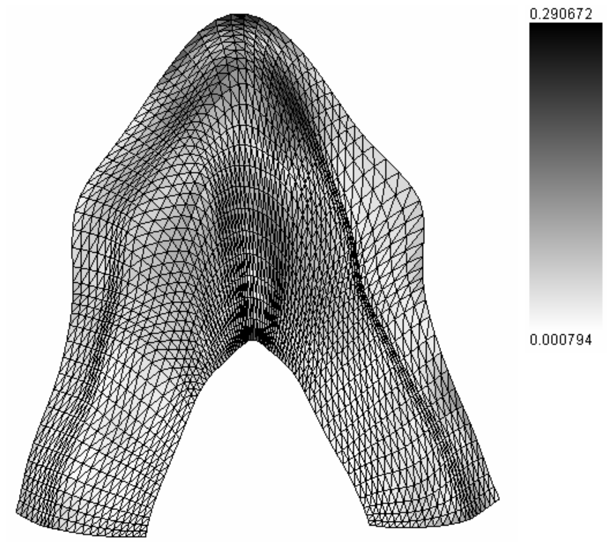

Fig. 7. *Planar development of a shoe last using the modified LAE model.*

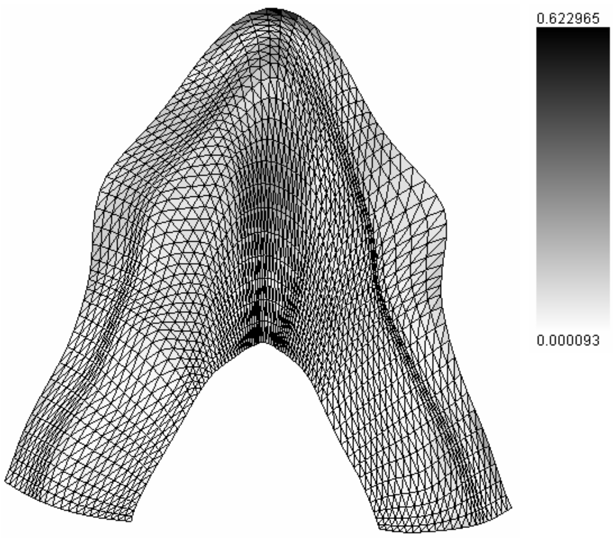

Fig. 8. *Planar development of a shoe last using the FEA method.*

Figure 7 illustrates the distortion of the planar development of the shoe last derived using the LAE model. In this case, the metric distortion is much lower than the previous example, since the surface of the last does not present areas with high-curvature. The parameter α was set to the default value 0.5. The high homogeneity index *h* guarantees homogeneous distribution of the metric distortion within the area of the planar development. Figure 8, displays the corresponding planar development obtained using FEA. In this case, the distortion of the triangles is larger – more than twice that of the previous development. Also, the homogeneity index is not as high as that produced by LAE. However, in this case the execution time for FEA was significantly lower than the first example. This is due to two reasons: the algorithm required fewer iterations to achieve convergence, and the triangles order was carefully designed as to minimize the bandwidth of the global stiffness matrix.

6 Summary

Classical Differential Geometry offers a host of results to analyze developability of free-form surfaces. However, none of these can be extended to the case of *digital surfaces*, i.e., surfaces defined approximately by a triangular mesh, replacing an exact analytic representation. Digital surface descriptions are gaining an increasing popularity as CAD/CAM and Computer Graphics applications are continuously shifting towards "digital modeling and processing". This chapter reviews related results from Differential Geometry and proceeds to developing quality control criteria for digital flattening (Section 3). Current flattening methods are analyzed in Section 4 and categorized on the basis of fundamental characteristics. Finally, in Section 5, two state-of-the-art methods for digital flattening are evaluated using realistic examples as well as the criteria/metrics of Section 3.

Although many flattening methods are constantly appearing, not much effort is directed towards "numerical quality control" of planar developments. The latter has been the focal point of this chapter, aiming at assisting practitioners in identifying the most appropriate method for each application.

References

1. Azariadis P., Aspragathos N. (1997). Design of Plane Patterns of Doubly Curved Surfaces, Computer-Aided Design, 29(10), 675-685.
2. Azariadis P., Aspragathos N. (2000). On Using Planar Developments to Perform Texture Mapping on Arbitrarily Curved Surfaces, Computers & Graphics, 24, 539-554.
3. Azariadis P., Aspragathos N. (2001). Geodesic Curvature Preservation In Surface Flattening Through Constrained Global Optimization, Computer-Aided Design, **33**(8), 581-591.
4. Azariadis P., Nearchou A, Aspragathos N. (2002). An evolutionary algorithm for generating planar developments of arbitrarily curved surfaces, Computers in Industry, **47**(3), 357-368.

5. Bodduluri R M C, Ravani B. (1993). Design of Developable Surfaces Using Duality Between Plane and Point Geometries, Computer-Aided Design, **25**(10), 621-632.
6. Bodduluri R M C, Ravani B. (1994), Geometric Design and Fabrication of Developable Bezier and B-spline Surfaces, Transactions of the ASME, Journal of Mechanical Design, Vol. 116/1043.
7. Chakib Bennis, Jean-Marc Vezien, Gerald Iglesias (1991). Piecewise Surface Flattening for Non-Distorted Texture Mapping, Computer Graphics, **25**(4), 237-246.
8. Charlie C.L. Wang, Shiang-Fong Chen, Matthew M.F. Yuen. (2001). Surface flattening for the fashion industry: a generic approach using spring-mass system, Computers in Industry 1548, 1-10.
9. Charlie C.L. Wang, Shiang-Fong Chen, Matthew M.F. Yuen. (2002). Surface flattening based on energy model, Computer-Aided Design, **34**, 823-833.
10. Do Carmo Manfredo (1976). Differential Geometry of Curves and Surfaces. Pren-tice-Hall, Englewood Cliffs, New Jersey.
11. E. L. Schwartz, A. Shaw, and E. Wolfson (1989). A numerical solution to the generalized mapmaker's problem: Flattening nonconvex polyhedral surfaces. IEEE Trans. on Pattern Analysis and Machine Intelligence, **11**(9), 1005-1008
12. E. Wolfson and E. L. Schwartz. (1989). Computing minimal distances on polyhedral surfaces. IEEE Trans. on Pattern Analysis and Machine Intelligence, **11**(9), 1001-1005.
13. Gershon Elber (1995). Model Fabrication Using Surface Layout Projection, Computer-Aided Design, **27**(4), 283-291.
14. Guoxin Yu, Nicholas M. Patrikalakis, Takashi Maekawa (2000). Optimal development of doubly curved surfaces, Computer Aided Geometric Design, **17**, 545-577.
15. Hinds B K, McCartney L, G Woods (1991). Pattern Development for 3D Surfaces, Computer-Aided Design, **23**(8), 583-592.
16. J. W. Sammon (1969). A nonlinear mapping for data structure analysis. IEEE Transactions on Computers, **18**(5), 401-409
17. Jerome Maillot, Hussein Yahia, Anne Verroust (1993). Interactive Texture Mapping, Proc. SIGGRAPH 93, Anaheim, California, 27-34 (1-6 August 1993).
18. Josef Hoschek (1998). Approximation of Surfaces of Revolution by Developable Surfaces, Computer-Aided Design, **30**(10), 757-763
19. Karger, A., Novak, J. (1985). Space Kinematics and Lie Groups, Gordon & Breach Science Publishers
20. Leopoldseder, S., Pottmann, H., Approximation of Developable Surfaces With Cone Spline Surfaces, Computer-Aided Design, **30**(7), 571-582
21. Lipschutz, M.M. (1969). Differential Geometry, McGraw-Hill, USA
22. Manning J R (1980). Computerized Pattern Cutting, Computer-Aided Design, **12**(1), 43-47.
23. McCartney, J., Hinds, B.K., Seow, B.L.(1999), The Flattening of triangulated Surfaces Incorporating Darts and Gussets, Computer-Aided Design, 31, 249-260.
24. Parida, L., Mudur, S.P. (1993), Constraint-Satisfying Planar Development of Complex Surfaces, Computer-Aided Design, **25**(4), 225-232
25. Redont P.(1989), Representation and Deformation of Developable Surfaces, Computer-Aided Design, **21**(1), 13-20

26. Sheffer, A. and de Sturler, E. (2000). Parameterization of faceted surfaces for meshing using angle based flattening. Engineering with Computers **17**(3) 326-337.
27. Sheffer, A. and de Sturler, E. (2002) Smoothing an Overlay Grid to Minimize Linear Distortion in Texture Mapping, ACM Transactions on Graphics, **21**(4), 874-890.
28. Shimada, T., Tada, Y. (1989). Development of curved surface using finite element method, Proc. 1st Int. Conf. Computer-Aided Optimum Design of Structures, Recent Advances, Springer-Verlag, 23-30.
29. Shimada, T., Tada, Y. (1991). Approximate Transformation of an Arbitrary Curved Surface Into a Plane Using Dynamic Programming, Computer-Aided Design, **23**(2), 153-159.
30. Song De Ma and Hiong Lin (1998). Optimal Texture Mapping, Proc. EUROGRAPHICS 88, 421-428.
31. Sundar, P., Varada, R. (1995). Evolution of generic Mathematical Models and Algorithms for the Surface Development and manufacture of Complex Ducts, Journal of Engineering for Industry, Vol.117, 177-185
32. Willmore, T.J., (1972). An Introduction to Differential Geometry, Oxford University Press.
33. Zigelman Gil, Kimmel Ron, Kiryati Nahum (2002). Texture Mapping using Surface Flattening via Multi-Dimensional Scaling. IEEE Transactions on Visualization and Computer Graphics, 8(2), 198-207.