Pseudomonotone Maps: Properties and Applications

NICOLAS HADJISAVVAS
Department of Product and System Design
Engineering, University of the Aegean,
Hermoupolis, Greece

MSC2000: 49J40; 49J53; 47H05; 47H04; 26B25

Article Outline

Introduction/Background

Pseudomonotone maps were introduced by Karamardian as a generalization of monotone maps [22]. Other generalizations include various kinds of pseudomonotone maps (quasimonotone, strictly quasimonotone, semistrictly quasimonotone . . .). These generalizations, the relations between them, as well as the relation to generalized convexity are discussed in other articles of the Encyclopedia (see ► Generalized monotone single valued maps and ► Generalized monotone multivalued maps and in [16].

It should be noted that the same term a "pseudomonotone map" has been introduced by Brezis to denote a totally different class of maps [4]. The main difference between the two classes is that pseudomonotone maps in the sense of Brézis are defined through a kind of continuity property, whereas Karamardian used only the order relation of real numbers in his definition. For this reason some authors, starting by Gwinner [12], use the term "topologically pseudomonotone" for pseudomonotone maps in the Brézis sense. Although it is possible to give a definition that includes both kinds of pseudomonotonicity [11], we will use the term "pseudomonotone" only in the sense defined by Karamardian.

Pseudomonotone maps have the advantage that they lead to generalizations of existence theorems for the Stampachia variational inequality problem (VIP), without imposing additional assumptions, and with practically the same proof as for monotone maps [16]. However, this quasi-identical treatement of the VIP is not extended to other topics, and the properties of the two classes of maps are often quite dissimilar. For instance, while the sum of two monotone maps is monotone, this is false for pseudomonotone maps. A vast theory has been developed for monotone maps, based on the concept of maximal monotonicity. By contrast, until recently it was believed that maximality plays no rôle for pseudomonotone maps. Consequently, some algorithms for finding the solution of VIP with maximal monotone maps have no extension to the pseudomonotone case.

This article will present some recent developments that can be considered as a first step towards filling the lacunae in the theory of pseudomonotone maps. In particular, maximal pseudomonotonicity will be discussed. The main tool is the definition of an equivalence relation in the set of all pseudomonotone maps. Also, a generalization of paramonotone maps and their use in cutting plane algorithms will be described. Finally, recent results on pseudoaffine maps and on the relation to monotone maps will be presented.

Definitions

Let X be a real Banach space and X^* be its dual. Given $x, y \in X$, $[x, y]$ denotes the line segment $\{(1-t)x + ty : t \in [0,1]\}$. For $K \subseteq X^*$, $\mathbb{R}_{++}K$ will be the set $\bigcup_{t>0} tK$. A multivalued map $T : X \to 2^{X^*}$ is a map whose values are subsets of X^*, possibly empty. The *domain* $D(T)$ of T is the set $\{x \in X : T(x) \neq \emptyset\}$, its graph the set $\mathrm{gr}(T) = \{(x, x^*) \in X \times X^* : x^* \in T(x)\}$ and its set of zeros is the set $Z_T = \{x \in X : 0 \in T(x)\}$. The map T is called upper sign-continuous [13] if for all $x \in D(T)$ and $v \in X$, the following implication holds:

$$\left(\forall t \in (0,1), \inf_{x^* \in T(x+tv)} \langle x^*, v\rangle \geq 0\right) \Rightarrow \sup_{x^* \in T(x)} \langle x^*, v\rangle \geq 0.$$

If T is upper hemicontinuous (i. e., its restriction on line segments is upper semicontinuous with respect to the weak* topology in X^*), then it is upper sign-continuous.

The map T is called monotone if for every (x, x^*), $(y, y^*) \in \mathrm{gr}(T)$, $\langle y^* - x^*, y - x\rangle \geq 0$; it is called maximal monotone if its graph is not strictly contained in the graph of any other monotone map. Also, T is called D-maximal monotone if its graph is not strictly contained in the graph of any other monotone map with the same domain.

The map T is called pseudomonotone if for every (x, x^*), $(y, y^*) \in \mathrm{gr}(T)$, the following implication holds.

$$\langle x^*, y - x\rangle \geq 0 \Rightarrow \langle y^*, y - x\rangle \geq 0.$$

Obviously, every monotone map is pseudomonotone.

Given a locally Lipschitz function $f : X \to \mathbb{R} \cup \{+\infty\}$, we denote by $\partial^0 f$ its Clarke subdifferential [7]. The locally Lipschitz function f is called pseudoconvex if for every $x \in \mathrm{dom}(f)$ and $x^* \in \partial^0 f(x)$ the following implication holds:

$$\langle x^*, y - x\rangle \geq 0 \Rightarrow f(y) \geq f(x).$$

It is known that a locally Lipschitz function f is pseudoconvex if and only if $\partial^0 f$ is a pseudomonotone map [23].

Formulation

Maximal Pseudomonotonicity

In order to introduce maximal pseudomonotone maps, one first defines an equivalence relation in the set of pseudomonotone maps. Two pseudomonotone maps T_1 and T_2 are called equivalent if they have the same domain, the same set of zeros, and for each x which is not a common zero, the elements of $T_1(x)$ are positive multiples of the elements of $T_2(x)$ and vice versa. In other words,

(a) $D(T_1) = D(T_2)$

(b) $Z_{T_1} = Z_{T_2}$,

(c) for every $x \in D(T_1)$, $\mathbb{R}_{++}T_1(x) = \mathbb{R}_{++}T_2(x)$.

In this case we write $T_1 \sim T_2$. This is an equivalence relation. Another aspect of this equivalence is provided by the Stampacchia variational inequality. Given a map T and a convex subset K of X, we denote by $S(T, K)$ the set of all $x \in K$ which are solutions of the VIP:

$$\forall y \in K, \exists x^* \in T(x) : \langle x^*, y - x\rangle \geq 0.$$

The following result holds [14].

Proposition 1 *Let T_1, T_2 be pseudomonotone maps. If $T_1 \sim T_2$, then $S(T_1, K) = S(T_2, K)$ for every convex set $K \subseteq X$. Conversely, if $S(T_1, K) = S(T_2, K)$ for every line segment K and T_1, T_2 have weak*-compact convex values, then $T_1 \sim T_2$.*

Since all equivalent maps provide the same solutions to VIP, one can choose any element of the equivalence class to study or even find the solutions.

Given a pseudomonotone map T, its equivalence class has a maximum with respect to graph inclusion. This is simply the map $\hat{T}$ defined by $\hat{T}(x) = \cup_{S \sim T} S(x)$ for all $x \in D(T)$. It can be shown [13] that $\hat{T}$ is also given by the formula

$$\hat{T}(x) = \begin{cases} \emptyset, & \text{if } x \notin D(T) \\ \mathbb{R}_{++}T(x), & \text{if } x \in D(T)\setminus Z_T \\ N_{L_{T,x}}, & \text{if } x \in Z_T \end{cases}$$

where $N_{L_{T,x}}$ is the normal cone at x to the set $L_{T,x} = \{y \in X : \exists y^* \in T(y), \langle y^*, y - x\rangle \leq 0\}$.

A pseudomonotone map $\hat{T}$ is called D-maximal pseudomonotone, if the graph of $\hat{T}$ is not properly contained in the graph of any other map with the same domain. When the domain of T is convex, there is an equivalent, more appealing definition for D-maximal pseudomonotonicity [14]:

Proposition 2 *Let T be pseudomonotone and such that $D(T)$ is convex. Then T is D-maximal pseudomonotone if, and only if, every pseudomonotone extension of T with the same domain is equivalent to T.*

Some properties of the set of zeros of T are provided by the following proposition [14].

Proposition 3 *Let T be D-maximal pseudomonotone. Then Z_T is weakly closed in $D(T)$. If in addition $D(T)$ is convex, then Z_T is also convex, and $z \in Z_T$ is equivalent to*

$$\forall (y, y^*) \in \mathrm{gr}(T), \langle y^*, y - z\rangle \geq 0.$$

The following proposition provides a simple criterion for showing the D-maximal pseudomonotonicity of a map [13].

Proposition 4 *Assume that T is pseudomonotone, upper-sign continuous, with weak*-compact, convex values and open domain $D(T)$. Then T is D-maximal pseudomonotone.*

A simple consequence of the above proposition is:

Corollary 5 *The Clarke subdifferential $\partial^0 f$ of a locally Lipschitz, pseudoconvex function $f : X \to \mathbb{R} \cup \{+\infty\}$ is a D-maximal pseudomonotone map.*

As was explained before, from the point of view of VIP one can use any element of the equivalence class. This is fortunate, since in many cases instead of showing that a D-maximal pseudomonotone map has a "nice" property (as is the case with maximal monotone maps), one shows that an equivalent map has this property. For instance, one has:

Proposition 6 *If T is D-maximal pseudomonotone, then $\hat{T}(x)$ is convex for every $x \in D(T)$. If in particular the assumptions of Proposition 4 are satisfied, then $\hat{T}(x) \cup \{0\}$ is weak*-closed.*

Here is a case where one can find an equivalent map with a better continuity property [13]:

Proposition 7 *Let $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a pseudomonotone map, upper sign-continuous, with compact convex values. If $D(T)$ is open and convex, then there exists an equivalent upper semicontinuous map T_1 with closed compact values.*

For instance let T be a single-valued pseudomonotone map defined on an open convex subset of $\mathbb{R}^n$. If T is hemicontinuous (i. e., continuous along line segments) then the above proposition guarantees that there exists an equivalent map which is continuous. Likewise, one can show that under some fairly general assumptions, T is equivalent to a map which is generically single-valued (i. e., is single valued except on a set of the first category). See Corollary 3.10 in [13].

A Generalization of Paramonotone Maps

A multivalued map T is called paramonotone if for every (x, x^*), $(y, y^*) \in \operatorname{gr}(T)$, $\langle y^* - x^*, y - x\rangle = 0$ implies that $x^* \in T(y)$ and $y^* \in T(x)$. It can be shown that the subdifferential of a proper lsc convex function is paramonotone [5]. Other examples of paramonotone maps are given in [21]. Paramonotone maps have been extensively used in algorithms for the solution of VIP [5,6,24]. The main reason is that these maps have the following "cutting plane property":

$$\left.\begin{array}{c} x \in S(T,K) \\ y \in K \\ \langle y^*, x - y\rangle \geq 0 \\ \text{for some } y^* \in T(y) \end{array}\right\} \Rightarrow y \in S(T,K). \qquad (1)$$

Assume that a map has property (1). If at the nth iteration of an algorithm one finds a point y_n that is not a solution of VIP, then all solutions of VIP belong to the intersection of K with the halfspace $\{x \in X : \langle y_n^*, x - y_n\rangle < 0\}$ where y_n^* is an arbitrary element of $T(y_n)$.
Let $K \subseteq X$ be nonempty, closed and convex. A single valued pseudomonotone map $T : K \to X^*$ is called pseudomonotone$_*$ if for all $x, y \in K$,

$$\langle T(x), y - x\rangle = \langle T(y), y - x\rangle = 0$$

implies that $T(x) = kT(y)$, for some $k > 0$ [8]. Note that single-valued pseudomonotone$_*$ maps are a generalization of single-valued paramonotone maps. To extend this generalization to the multivalued case, one needs the tools presented in the previous subsection.

Definition 8 [17] A map $T : X \to 2^{X^*}$ is pseudomonotone$_*$ on K if it is pseudomonotone and for every $x, y \in K$ and $x^* \in T(x)$, $y^* \in T(y)$, $\langle x^*, y - x\rangle = \langle y^*, y - x\rangle = 0$ imply $x^* \in \hat{T}(y)$ and $y^* \in \hat{T}(x)$.

It is easy to see that every paramonotone map is pseudomonotone$_*$. Other classes of pseudomonotone$_*$ maps is provided by the following propositions [17].

Proposition 9 *The Clarke subdifferential $\partial^0 f$ of a locally Lipschitz pseudoconvex function f is pseudomonotone$_*$.*

Proposition 10 *If the map T is pseudomonotone$_*$, then any map equivalent to T is pseudomonotone$_*$.*

Proposition 9 is a particular case of a more general situation. A map T is called cyclically pseudomonotone [9,10] if for every $(x_i, x_i^*) \in \operatorname{gr}(T)$, $i = 1, 2, \ldots, n$, the following implication holds:

$$\langle x_i^*, x_{i+1} - x_i\rangle \geq 0,\ \forall i = 1, 2, \ldots, n-1 \\ \Rightarrow \langle x_n^*, x_1 - x_n\rangle \leq 0.$$

Proposition 11 *If T is D-maximal pseudomonotone and cyclically pseudomonotone with convex domain, then it is pseudomonotone$_*$.*

Since the Clarke subdifferential of a locally Lipscitz pseudoconvex function is D-maximal pseudomonotone and cyclically pseudomonotone, we see that the previous proposition implies Proposition 9.
We saw that paramonotone maps have the cutting plane property (1). The same is true for pseudomonotone$_*$ maps; what is more interesting is that these maps are characterized in some sense by the cutting plane property:

Proposition 12 *Let T be pseudomonotone on the convex set K. If T is pseudomonotone$_*$, then property (1) holds on every subset of K. Conversely, if property (1) holds on every convex, compact subset of K and T has convex, weak*-compact values, then T is pseudomonotone$_*$ on the interior of K.*

If T is single-valued, the assumption of pseudomonotonicity becomes redundant:

Proposition 13 *Let $T : K \to X^*$ be hemicontinuous. If T has property (1) on each convex compact subset of K, then T is pseudomonotone on K and pseudomonotone$_*$ on its interior.*

In Sect. "Methods/Applications" we will show how to apply pseudomonotone$_*$ maps for the solution of variational inequalities.

Pseudoaffine Maps

Given a convex subset K of $\mathbb{R}^n$, a single-valued map $T : K \to \mathbb{R}^n$ is called pseudoaffine (or PPM, as in [3]) if both T and $-T$ are pseudomonotone. These maps were studied in [3] in connection with VIP. It is easy to see that a differentiable function $f : K \to \mathbb{R}$ is pseudolinear (i. e., both f and $-f$ are pseudoconvex) if and only if ∇f is pseudoaffine. It is not hard to show that pseudolinear functions defined on the whole space $\mathbb{R}^n$ have a very particular form [2,25]:

Proposition 14 *A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is pseudolinear if and only if there exist a vector $u \in \mathbb{R}^n$ and a one-variable differentiable function h whose derivative is always positive or identical to zero, such that $f(x) = h(\langle u, x\rangle)$.*

If $T = \nabla f$ in this case, then $T(x) = h'(u, x)$, i. e., T is equal to a positive multiple of a constant vector. For general pseudoaffine maps (i. e., those that are not necessarily equal to a gradient) that are defined on the whole space, the following elegant characterization has been shown:

Proposition 15 *A map $T : \mathbb{R}^n \to \mathbb{R}^n$ is pseudoaffine if and only if there exists a positive function $g : \mathbb{R}^n \to \mathbb{R}$, a skew-symmetric linear map A and a vector u such that*

$$\forall x \in \mathbb{R}^n,\ T(x) = g(x)(Ax + u).$$

The proof of the above result needs some "global" arguments provided by algebraic topology and by projective geometry [2].

Pseudomonotone vs. Monotone Maps

One of the basic differences between the class of monotone maps and the class of pseudomonotone maps has to do with their stability with respect to some operations. For instance, the class of monotone maps is stable with respect to addition (i. e., the sum of two monotone maps is monotone), while this is not the case for pseudomonotone maps. By contrast, the product of a pseudomonotone map with a positive function produces a pseudomonotone map while this is not the case for monotone maps.
In particular, it was noted in [1] that a map $T : X \to 2^{X^*}$ is monotone if and only if for every $x^* \in X^*$ the map $T + x^*$ is pseudomonotone. More recently He [18] and Isac and Motreanu [20] obtained another result in this direction. Assume that X is a Hilbert space (in [18] one considered $X = \mathbb{R}^n$) and $K \subseteq X$ is a convex set with nonempty interior. Let further $T : K \to X^*$ be a continuous single-valued map which is Gâteaux differentiable in the interior of K. Then T is monotone if and only if $T + x^*$ is pseudomonotone for all x^* in a straight line of X^*. The differentiability assumption is essential in the argument of both papers [18,20] because the proof is based on a first-order characterization of generalized monotonicity.
In a recent paper [15] it was shown that the differentiability assumption is redundant, and one can also weaken considerably the assumption that the interior of K is nonempty. Given $x^* \in X^*$ and a set $K \subseteq X$, one says that x^* is perpendicular to K if the value of x^* is constant on K, i. e., $\langle x^*, y - x\rangle = 0$ for all $x, y \in K$. The following proposition holds.

Proposition 16 *Let $K \subseteq X$ be nonempty and convex and $T : K \to 2^{X^*}$ be a map with nonempty* *values. Assume that there exists a straight line* $S = \{x_0^* + tx^* : t \in \mathbb{R}\}$ *in* X^* *such that* x^* *is not perpendicular to* K*, and for all* $z^* \in S$*,* $T + z^*$ *is pseudomonotone. Then* T *is monotone.*

In case K has nonempty interior or, more generally, nonempty quasi-interior, the assumption "x^* is not perpendicular to K" is automatically fulfilled. It should also be noted that the results of this subsection are also true if we replace "pseudomonotone" by "quasimonotone" (see the article ► Generalized monotone multivalued maps in this Encyclopedia for the definition).

Methods/Applications

Many of the algorithms used to find a solution of a variational inequality with a paramonotone map, can be also used in the more general case of a pseudomonotone$_*$ map. We illustrate this by an example of a perturbed auxiliary problem method. Let K be a closed convex subset of a Hilbert space H, $T : K \to 2^H$ a map with nonempty values. Choose a Gâteaux differentiable strongly convex function $M : H \to \mathbb{R}$ with a weakly continuous derivative (we can take for instance $M(x) = \|x\|^2/2$). Construct a sequence $\{x_k\}_{k\in\mathbb{N}}$ by the following algorithm.

(i) Choose an arbitrary $x_0 \in K$.
(ii) Having chosen x_k, find $x_{k+1} \in K$ and $x^*_{k+1} \in T(x_{k+1})$ such that

$$\forall y \in K,\ \langle \mu_k x^*_{k+1} + M'(x_{k+1}) - M'(x_k),\ y - x_{k+1} \rangle \geq 0$$

where $\{\mu_k\}_{k\in\mathbb{N}}$ is a sequence of positive constants bounded from below. Note that finding x_{k+1} amounts to solving VIP for the perturbed map $T_{k+1}(\cdot) = \mu_{k+1} T(\cdot) + M'(\cdot) - M'(x_k)$. This problem can be much easier than the original one, since for instance if T is weakly monotone and μ_{k+1} is small, then T_{k+1} is strongly monotone. Assume that VIP has a solution and that the sequence $\{x_k\}_{k\in\mathbb{N}}$ is well-defined. Then it can be shown that if T is pseudomonotone$_*$ and satisfies a fairly general continuity condition, then the sequence $\{x_k\}_{k\in\mathbb{N}}$ converges weakly to a solution of VIP for T. Details can be found in [17].

Conclusions

The theory of pseudomonotone maps is far from been developed to a satisfactory level. By contrast, the theory of monotone maps has reached a high level of maturity [19]. It is hoped that some of the recent advances presented here, and in particular the ideas on maximality of pseudomonotone maps, will provide a firm background for the study of pseudomonotone maps. This is illustrated by the ease and naturalness with which notions like paramonotonicity can be generalized to pseudomonotone maps, a task that seemed almost impossible before the introduction of maximal pseudomonotone maps. In addition, the new notion of a pseudomonotone$_*$ map seems to be ideally fit the cutting plane property (see Proposition 12) and this adds some confidence that the definition of maximality is on the right way.

References

1. Aussel D, Corvellec JN, Lassonde M (1994) Subdifferential characterization of quasiconvexity and convexity. J Convex Anal 1:195–201
2. Bianchi M, Hadjisavvas N, Schaible S (2003) On Pseudomonotone Maps T for which $-T$ is also pseudomonotone. J Convex Anal 10:149–168
3. Bianchi M, Schaible S (2000) An extension of pseudolinear functions and variational inequality problems. J Optim Theory Appl 104:59–71
4. Brezis H (1968) Equations et inéquations nonlinéaires dans les espaces vectoriels en dualité. Ann Inst Fourier 18:115–175
5. Bruck RE Jr (1976) An iterative solution of a variational inequality for certain monotone operators in Hilbert space. Bull Amer Math Soc 81:890–892; Corrigendum, Bull Amer Math Soc 82 (1976) 353
6. Burachik R, Iusem A (1998) A generalized proximal point algorithm for the variational inequality problem in Hilbert space. SIAM J Optim 8:197–216
7. Clarke FH (1983) Optimization and nonsmooth Analysis. Wiley Interscience, CE2
8. Crouzeix JP, Marcotte P, Zhu D (2000) Conditions ensuring the applicability of cutting-plane methods for solving variational inequalities, Math Program 88:521–539
9. Daniilidis A, Hadjisavvas N (1999) On the subdifferentials of pseudoconvex and quasiconvex functions and cyclic monotonicity, J Math Anal Appl 237:30–42
10. Daniilidis A, Hadjisavvas N (2000) On generalized cyclically monotone operators and proper quasimonotonicity, Optim 47:123–135
11. Domokos A, Kolumban J (2000) Comparison of two different types of pseudomonotone mappings, In: Popoviciu E (ed) Seminaire de la théorie de la meilleure approximation, convexité et optimisation. CE3 pp 95–103
12. Gwinner J (1981) On fixed points and variational inequalities—a circular tour. Nonlinear Anal 5:565–583
13. Hadjisavvas N (2003) Continuity and maximality properties of pseudomonotone operators. J Convex Anal 10:459–469

CE2 Please provide publisher location.
CE3 Please provide publisher and publisher location.

Editor's or typesetter's annotations (will be removed before the final TEX run)

14. Hadjisavvas N (2003) Maximal pseudomonotone operators, In: Crespi GP, Guerraggio A, Miglierina E, Rocca M (eds) Recent advances in Optimization. Datanova Editrice, CE2 , pp 87–100
15. Hadjisavvas N (2006) Translations of quasimonotone maps and monotonicity. Appl Math Lett 19:913–915
16. Hadjisavvas N, Komlosi S, Schaible S (2005) Handbook of generalized convexity and generalized monotonicity. Springer, CE2
17. Hadjisavvas N, Schaible S (2006) On a generalization of paramonotone maps and its application to solving the Stampacchia variational inequality. Optim 55:593–604
18. He Y (2004) A relationship between pseudomonotone and monotone mappings. Appl Math Lett 17:459–461
19. Hu S, Papageorgiou NS (1997) Handbook of Multivalued Analysis, vols I, II. Kluwer Academic Publishers, CE2
20. Isac G, Motreanu D (2004) Pseudomonotonicity and quasimonotonicity by translations versus monotonicity in Hilbert spaces. Austral J Math Anal Appl 1:1–8
21. Iusem AN (1998) On some properties of paramonotone operators. J Convex Anal 5:269–278
22. Karamardian S (1976) Complementarity over cones with monotone and pseudomonotone maps. J Optim Theory Appl 18:445–454
23. Penot J-P, Quang PH (1997) Generalized convexity of functions and generalized monotonicity of set-valued maps. J Optim Theory Appl 92:343–356
24. Solodov MD, Svaiter BF (2000) An inexact hybrid generalized proximal point algorithm and some new results on the theory of Bregman functions. Math Oper Res 25:214–230
25. Thompson WA, Parke DW (1973) Some properties of generalized concave functions. Oper Res 21:305–313