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Local Boundedness Properties for Generalized Monotone Operators

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We show that a well-known property of monotone operators, namely local boundedness in the interior of their domain, remains valid for the larger class of premonotone maps. This generalizes a similar result by Iusem, Kassay and Sosa (J. Convex Analysis 16 (2009) 807–826) to Banach spaces.

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1. Introduction

In recent years, operators which have some kind of generalized monotonocity property have received a lot of attention (see for example [7] and the references therein). Many papers considering generalized monotonicity were devoted to the investigation of its relation to generalized convexity; others studied the existence of solutions of generalized monotone variational inequalities and, in some cases, derived algorithms for finding such solutions.

Monotone operators are known to have many very interesting properties. For instance, it is known that a monotone operator T defined on a Banach space is locally bounded in the interior of its domain. Actually by the Libor Veselý theorem, whenever T is maximal monotone and $\overline{D(T)}$ is convex, interior points of the domain $D(T)$ are the only points of $\overline{D(T)}$ where T is locally bounded. So the question nat-

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urally arises: are these properties shared by other operators which satisfy a more relaxed kind of monotonicity?

In a recent paper, Iusem, Kassay and Sosa [8] introduced the class of the so-called premonotone operators. This class includes monotone operators, but contains many more: for example, if T is monotone and R is globally bounded, then $T + R$ is premonotone. In fact, it includes ε-monotone operators which are related to the very useful ε-subdifferentials [9, 10]. In [8] it is shown that, in a finite dimensional space, premonotone operators are locally bounded in the interior of their domain. The proof was based on the finite dimensionality of the space.

In this paper we will show that these results remain valid in infinite dimensional Banach spaces. We also show that some properties of monotone operators remain valid in a much more general context. More precisely, after some preliminary definitions and results in Section 2, we introduce in Section 3 the notion of a premonotone bifunction. We will show that such bifunctions are locally bounded in the interior of their domain and we will deduce local boundedness of premonotone operators. We will also show a generalization of the Libor Veselý theorem.

Let us fix some notation and recall some definitions. In what follows, X will be a Banach space. Given a multivalued operator $T : X \to 2^{X^*}$, we will set $D(T) = \{x \in X : T(x) \neq \emptyset\}$ and $\mathrm{gr}(T) = \{(x, x^*) \in X \times X^* : x^* \in T(x)\}$ to be its domain and its graph, respectively. T is called norm×weak* closed (resp., sequentially norm×weak* closed) if $\mathrm{gr}(T)$ is closed (resp., sequentially closed) in the norm×weak* topology of $X \times X^*$. Given $x_0 \in X$, T is called sequentially norm×weak* closed at x_0 if for every sequence $(x_n, x_n^*) \in \mathrm{gr}(T)$ such that $x_n \to x_0$ and x_n^* weak*-converges to some $x_0^* \in X^*$, one has $x_0^* \in T(x_0)$. T is called monotone if $\langle x^* - y^*, x - y\rangle \geq 0$ for all $(x, x^*), (y, y^*) \in \mathrm{gr}(T)$. It is called maximal monotone if it is monotone, and for each monotone operator S such that $\mathrm{gr}(T) \subseteq \mathrm{gr}(S)$, one has $T = S$. The operator T is called locally bounded at $x_0 \in X$ if there exists some neighborhood V of x_0 such that the set $T(V) := \bigcup_{x\in V} T(x)$ is bounded.

Given a convex set $C \subseteq X$ and $x \in C$ we will denote by $N_C(x)$ the normal cone of C at x:

$$N_C(x) = \{x^* \in X^* : \forall y \in C, \langle x^*, y - x\rangle \leq 0\}.$$

2. σ-Monotone operators

Most definitions and many of the results of this section are essentially due to [8], the main difference being that in [8] one considers premonotone operators in $\mathbb{R}^n$, without specifying a given σ.

Definition 2.1. (i) Given an operator $T : X \to 2^{X^*}$ and a map $\sigma : D(T) \to \mathbb{R}_+$, T is said to be *σ-monotone* if for every $x, y \in D(T)$, $x^* \in T(x)$ and $y^* \in T(y)$,

$$\langle x^* - y^*,\ x - y\rangle \geq -\min\{\sigma(x), \sigma(y)\}\|x - y\|. \tag{1}$$

(ii) An operator T is called premonotone if it is *σ-monotone* for some $\sigma : D(T) \to \mathbb{R}_+$.

(iii) A σ-monotone operator T is called *maximal σ-monotone*, if for every operator T' which is σ'-monotone with $\mathrm{gr}(T) \subseteq \mathrm{gr}(T')$ and σ' an extension of σ, one has $T = T'$.

The notion of premonotone operators for the finite-dimensional case is introduced in [8]. The same paper also contains examples of maximal σ-monotone operators.

Remark 2.2. (i) It should be noticed that $T : X \to 2^{X^*}$ is σ-monotone if and only if

$$\forall x, y \in D(T), \ x^* \in T(x), \ y^* \in T(y), \ \langle x^* - y^*, \ x - y\rangle \geq -\sigma(y)\|x - y\|. \quad (2)$$

(ii) If $\sigma(y) = 2\varepsilon \geq 0$ for each $y \in D(T)$, then T is called ε-monotone [10].

(iii) Definition 2.1 does not allow negative values for σ, since this can only happen in very special cases. For instance, if T satisfies (1) and its domain contains any line segment $[x_0, y_0] := \{(1-t)x_0 + ty_0 : t \in [0, 1]\}$, then the set of points $x \in [x_0, y_0]$ where $\sigma(x) < 0$ is at most countable. Indeed, if this is not the case, then there exists $\varepsilon > 0$ such that $\sigma(x) < -\varepsilon$ for infinitely many $x \in [x_0, y_0]$. Choose $x_0^* \in T(x_0)$, $y_0^* \in T(y_0)$. Given $n \in \mathbb{N}$, choose $x_k = x_0 + t_k(y_0 - x_0)$, $k = 1, \ldots n - 1$, such that $0 < t_1 < \cdots < t_{n-1} < 1$ and $\sigma(x_k) < -\varepsilon$. Then choose $x_k^* \in T(x_k)$. Set $x_n = y_0$ and $x_n^* = y_0^*$. Relation (2) gives for all $k = 0, 1, \ldots n - 1$:

$$\left\langle x_{k+1}^* - x_k^*, x_{k+1} - x_k \right\rangle \geq \varepsilon \left\| x_{k+1} - x_k \right\| \Rightarrow$$
$$\left\langle x_{k+1}^* - x_k^*, y_0 - x_0 \right\rangle \geq \varepsilon \left\| y_0 - x_0 \right\|.$$

Adding these inequalities for $k = 0, 1, \ldots n-1$ yields $\langle y_0^* - x_0^*, y_0 - x_0\rangle \geq n\varepsilon\|y_0 - x_0\|$. This should hold for each $n \in \mathbb{N}$, which is impossible.

(iv) The notion of premonotonicity is not suited to linear operators, since every σ-monotone linear operator $T : X \to X^*$ is in fact monotone. Indeed, in this case putting $y = 0$ in (2) we find

$$\forall x \in X, \quad \langle Tx, x\rangle \geq -\sigma(0) \|x\|. \quad (3)$$

Replacing x by $x - y$ we deduce that

$$\forall x, y \in X, \quad \langle Tx - Ty, x - y\rangle \geq -\sigma(0) \|x - y\|.$$

Thus T is ε-monotone with $\varepsilon = \sigma(0)/2$. Then Proposition 3.2 in [10] shows that T is monotone.

(v) A σ-monotone operator is maximal σ-monotone if and only if, for every operator T' which is σ'-monotone with $\mathrm{gr}(T) \subseteq \mathrm{gr}(T')$ and $\sigma'(x) \leq \sigma(x)$ for all $x \in D(T)$, one has $T = T'$.

The following proposition is an easy consequence of Zorn's Lemma, as for monotone operators.

Proposition 2.3. *Every σ-monotone operator has a maximal σ-monotone extension.*

Definition 2.4. Let A be a subset of X. Given a mapping $\sigma : A \to \mathbb{R}_+$, two pairs $(x, x^*), (y, y^*) \in A \times X^*$ are *σ-monotonically related* if

$$\langle x^* - y^*, x - y \rangle \geq - \min\{\sigma(x), \sigma(y)\} \|x - y\|.$$

The proof of the following proposition is obvious.

Proposition 2.5. *The σ-monotone operator $T : X \to 2^{X^*}$ is maximal σ-monotone if and only if, for every point $(x_0, x_0^*) \in X \times X^*$ and every extension σ' of σ to $D(T) \cup \{x_0\}$ such that (x_0, x_0^*) is σ'-monotonically related to all pairs $(y, y^*) \in \mathrm{gr}(T)$, we have $(x_0, x_0^*) \in \mathrm{gr}(T)$.*

Given an operator $T : X \to 2^{X^*}$, we define the function $\sigma_T : D(T) \to \mathbb{R}_+ \cup \{\infty\}$ by

$\sigma_T(y) = \inf\{a \in \mathbb{R}_+ : \langle x^* - y^*, x - y \rangle \geq -a \|x - y\|, \ \forall (x, x^*) \in \mathrm{gr}(T), y^* \in T(y)\}.$

Note that if the operator T is premonotone, then

$$\sigma_T = \inf\{\sigma : T \text{ is } \sigma\text{-monotone}\}$$

and thus σ_T is finite, and T is σ_T-monotone. Also in this case, it is obvious that

$$\sigma_T(y) = \max \left\{ \sup\{ \frac{\langle x^* - y^*, y - x \rangle}{\|y - x\|} : x \in X \backslash \{y\}, \, x^* \in T(x), \, y^* \in T(y)\}, 0 \right\} \quad (4)$$

(see also [8]). The following result is due to [8].

Proposition 2.6. *Let an operator T be given.*

(i) *σ_T is finite and T is σ_T-monotone, if and only if T is σ-monotone for some σ.*

(ii) *σ_T is finite and T is maximal σ_T-monotone, if and only if T is maximal σ-monotone for some σ.*

Proof. We have only to prove that whenever T is maximal σ-monotone for some σ, then it is maximal σ_T-monotone. Assume that $S : X \to 2^{X^*}$ is σ'-monotone with $\mathrm{gr}(T) \subseteq \mathrm{gr}(S)$ and σ' an extension of σ_T. Since $\sigma' = \sigma_T \leq \sigma$ on $D(T)$, by Remark 2.2(v) we get that $S = T$. Hence, T is maximal σ_T-monotone. □

Proposition 2.7. *Every maximal σ-monotone operator T is convex-valued and weak* closed-valued. Moreover, if σ is defined and usc at some point $x_0 \in \overline{D(T)}$, then T is sequentially norm×weak* closed at x_0.*

Proof. Let $T : X \to 2^{X^*}$ be a maximal σ-monotone operator and $(x, x_1^*), (x, x_2^*) \in \mathrm{gr}(T)$, $\lambda \in [0, 1]$. Then for each $(y, y^*) \in \mathrm{gr}(T)$,

$$\begin{aligned} & \langle \lambda x_1^* + (1 - \lambda) x_2^* - y^*, x - y \rangle \\ = {} & \lambda \langle x_1^* - y^*, x - y \rangle + (1 - \lambda) \langle x_2^* - y^*, x - y \rangle \\ \geq {} & - \lambda \min\{\sigma(x), \sigma(y)\} \|x - y\| - (1 - \lambda) \min\{\sigma(x), \sigma(y)\} \|x - y\| \\ = {} & - \min\{\sigma(x), \sigma(y)\} \|x - y\|. \end{aligned}$$

That is, $(x, \lambda x_1^* + (1-\lambda)x_2^*)$ is σ-monotonically related with all $(y, y^*) \in \mathrm{gr}(T)$. Now, it follows from Proposition 2.5 that $(x, \lambda x_1^* + (1-\lambda)x_2^*) \in \mathrm{gr}(T)$ which implies that $T(x)$ is convex. Likewise, one can show that $T(x)$ is weak* closed.

We now show sequential closedness: suppose that (x_n, x_n^*) is a sequence in $\mathrm{gr}(T)$ such that $x_n \to x_0$ and $x_n^* \xrightarrow{w^*} x_0^*$. Assume that σ is usc at x_0. It follows from the σ-monotonicity of T that for each $(y, y^*) \in \mathrm{gr}(T)$ we have

$$\langle x_n^* - y^*, x_n - y\rangle \geq -\min\{\sigma(x_n), \sigma(y)\}\|x_n - y\|.$$

By taking limits in the above inequality and using the upper semicontinuity of σ at x_0 and the fact that $\{x_n^*\}$ is a bounded sequence, we get

$$\langle x_0^* - y^*, x_0 - y\rangle \geq -\min\{\sigma(x_0), \sigma(y)\}\|x_0 - y\|$$

which implies that (x_0, x_0^*) is σ-monotonically related with all $(y, y^*) \in \mathrm{gr}(T)$. By using Proposition 2.5 we deduce that $(x_0, x_0^*) \in \mathrm{gr}(T)$. □

We note that, as for monotone operators, in general $\mathrm{gr}(T)$ is only sequentially norm×weak* closed, not norm×weak* closed [5]. However, we will see in the next section that maximal σ-monotone operators are actually usc in the interior of their domains.

The assumption of upper semicontinuity of σ cannot be omitted from Proposition 2.7, as the following example shows. This is also an example of a premonotone operator which is not ε-monotone. Note that for $T : \mathbb{R} \to \mathbb{R}$ we have

$$\sigma_T(y) = \max\left\{\sup_{x\leq y}\{T(x) - T(y)\}\,,\ \sup_{x\geq y}\{T(y) - T(x)\}\right\}. \tag{5}$$

Example 2.8. We define the functions $\varphi, \sigma : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) = \begin{cases} x\sin^2 x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

and

$$\sigma(x) = \max\left\{\varphi(x), \max_{z\leq x}\varphi(z) - \varphi(x)\right\}.$$

We show that φ is σ-monotone, i.e., for all $x, y \in \mathbb{R}$ the following inequality holds:

$$(\varphi(x) - \varphi(y))\,(x - y) \geq -\min\{\sigma(x), \sigma(y)\}\,|x - y|\,.$$

We may assume without loss of generality that $x \leq y$, so we have to prove that $\varphi(x) - \varphi(y) \leq \min\{\sigma(x), \sigma(y)\}$. Indeed,

$$\varphi(x) - \varphi(y) \leq \varphi(x) \leq \sigma(x)$$

and

$$\varphi(x) - \varphi(y) \leq \max_{z\leq y}\varphi(z) - \varphi(y) \leq \sigma(y)$$

so φ is σ-monotone. Note that φ is not ε-monotone since $(\varphi(x)-\varphi(y))\operatorname{sgn}(x-y)$ is not bounded from below (take $y=2k\pi+\pi/2$, $x=2k\pi+\pi$ for large $k\in\mathbb{N}$).

We now change φ and σ at one point: define $T,\sigma_1:\mathbb{R}\to\mathbb{R}$ by

$$T(x)=\begin{cases}\varphi(x) & \text{if } x\neq\dfrac{\pi}{2},\\ \dfrac{\pi}{4} & \text{if } x=\dfrac{\pi}{2},\end{cases}\quad\text{and}\quad \sigma_1(x)=\begin{cases}\sigma(x) & \text{if } x\neq\dfrac{\pi}{2},\\ \dfrac{\pi}{4} & \text{if } x=\dfrac{\pi}{2}.\end{cases}$$

One can readily show that T is σ_1-monotone.

Now let $\widetilde{T}$ be a maximal σ_1-monotone extension of T. Its graph is not closed; indeed $(\pi/2,\pi/2)$ belongs to the closure of $\operatorname{gr}(\widetilde{T})$. However, it does not belong to $\operatorname{gr}(\widetilde{T})$ since it is not σ_1-monotonically related to $(\pi,0)\in\operatorname{gr}(\widetilde{T})$: since $\sigma_1(\pi)=\max_{z\leq\pi}\varphi(z)\geq\varphi(\pi/2)=\pi/2$, one has

$$\left(\frac{\pi}{2}-0\right)\operatorname{sgn}\left(\frac{\pi}{2}-\pi\right)=-\frac{\pi}{2}<-\frac{\pi}{4}=-\min\left\{\sigma_1\left(\frac{\pi}{2}\right),\sigma_1(\pi)\right\}.$$

3. Local boundedness and related properties

Some properties of σ-monotone operators can be more easily investigated through the use of σ-monotone bifunctions that we now introduce. Let X be a Banach space, C a nonempty subset of X and $\sigma:C\to\mathbb{R}_+$ be a map. A bifunction $F:C\times C\to\mathbb{R}$ will be called σ-monotone if

$$\forall x,y\in C,\quad F(x,y)+F(y,x)\leq\min\{\sigma(x),\sigma(y)\}\|x-y\|. \tag{6}$$

Equivalently, F is σ-monotone if

$$\forall x,y\in C,\quad F(x,y)+F(y,x)\leq\sigma(y)\|x-y\|. \tag{7}$$

This notion is a generalization of the notion of monotone bifunction introduced in [4], where σ is identically zero.

Given any bifunction $F:C\times C\to\mathbb{R}$, we define as in [1, 3] the operator $A^F:X\to 2^{X^*}$ by

$$A^F(x)=\begin{cases}\{x^*\in X^*:\forall y\in C, F(x,y)\geq\langle x^*,y-x\rangle\} & \text{if } x\in C,\\ \emptyset & \text{if } x\notin C.\end{cases}$$

Note that in case $F(x,x)=0$ for all $x\in C$, one has $A^F(x)=\partial F(x,\cdot)(x)$ (the subdifferential of the function $F(x,\cdot)$ at x).

Proposition 3.1. *For a σ-monotone bifunction F, A^F is σ-monotone.*

Proof. Let $x^*\in A^F(x)$ and $y^*\in A^F(y)$. By the definition of A^F,

$$F(x,y)\geq\langle x^*,y-x\rangle$$

and

$$F(y,x) \geq \langle y^*, x-y\rangle.$$

From these inequalities we obtain

$$\langle x^* - y^*, x-y\rangle \geq -F(x,y) - F(y,x) \geq -\min\{\sigma(x), \sigma(y)\}\|x-y\|.$$

□

Definition 3.2. A σ-monotone bifunction F is called *maximal σ-monotone* if A^F is maximal σ-monotone.

For a given operator $T : X \to 2^{X^*}$, as in [6] we define $G_T : D(T) \times D(T) \to \mathbb{R} \cup \{+\infty\}$ by $G_T(x,y) = \sup_{x^* \in T(x)} \langle x^*, y-x\rangle$. For each $x \in D(T)$, $G_T(x,\cdot)$ is lsc and convex, and $G_T(x,x) = 0$. The following result shows that G_T is actually real valued whenever T is σ-monotone, and establishes some relations between σ-monotonicity of G_T and T.

Proposition 3.3. *Let T be an operator. Then the following statements are true.*

(i) *If T is σ-monotone, then G_T is a real-valued, σ-monotone bifunction.*

(ii) *If T is maximal σ-monotone, then G_T is a maximal σ-monotone bifunction and $A^{G_T} = T$.*

(iii) *Suppose that T is σ-monotone with closed convex values and $D(T) = X$. If G_T is maximal σ-monotone, then T is maximal σ-monotone.*

Proof. (i) Let $T : X \to 2^{X^*}$ be σ-monotone. Given $x, y \in D(T)$, for every $x^* \in T(x)$ and $y^* \in T(y)$, we have

$$\langle x^* - y^*, x-y\rangle \geq -\sigma(y)\|x-y\|.$$

Thus

$$\langle y^*, x-y\rangle + \langle x^*, y-x\rangle \leq \sigma(y)\|x-y\|.$$

This implies that

$$\sup_{y^* \in T(y)} \langle y^*, x-y\rangle + \sup_{x^* \in T(x)} \langle x^*, y-x\rangle \leq \sigma(y)\|x-y\|.$$

Form here we conclude that

$$\forall x, y \in D(T), \quad G_T(x,y) + G_T(y,x) \leq \sigma(y)\|x-y\|.$$

Consequently, $G_T(x,y) \in \mathbb{R}$ for all $x, y \in D(T)$ and G_T is a σ-monotone bifunction.

(ii) Let $(x, z^*) \in \mathrm{gr}(T)$. For every $y \in C$ we have

$$G_T(x,y) = \sup_{x^* \in T(x)} \langle x^*, y-x\rangle \geq \langle z^*, y-x\rangle.$$

This means that $z^* \in A^{G_T}(x)$; i.e., $T(x) \subseteq A^{G_T}(x)$. It follows from Proposition 3.1 and part (i) that A^{G_T} is σ-monotone. Since T is maximal σ-monotone, we conclude that $T = A^{G_T}$.

(*iii*) Since G_T is maximal σ-monotone by assumption, A^{G_T} is maximal σ-monotone. Let $x \in X$ and $z^* \in A^{G_T}(x)$. Then

$$G_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq \langle z^*, y - x \rangle.$$

Now, the separation theorem implies that $z^* \in T(x)$. Thus, $\mathrm{gr}(A^{G_T}) \subseteq \mathrm{gr}(T)$. This implies that $T = A^{G_T}$ and T is maximal σ-monotone. □

Remark 3.4. Given a maximal σ-monotone bifunction F, according to Proposition 3.3, we can construct A^F and the σ-monotone bifunction $G := G_{A^F}$. One has $G(x, y) \leq F(x, y)$ for all $x, y \in D(A^F)$. It follows from Proposition 3.3 that $A^F = A^G$. However, Example 2.5 of [6] implies that the correspondence $F \mapsto A^F$ is not one to one, even for the monotone case $\sigma \equiv 0$.

We now generalize a definition from [6].

Definition 3.5. A bifunction $F : C \times C \to \mathbb{R}$ is called:

(i) Locally bounded at $(x_0, y_0) \in X \times X$ if there exist an open neighborhood V of x_0, an open neighborhood W of y_0 and $M \in \mathbb{R}$ such that $F(x, y) \leq M$ for all $(x, y) \in (V \times W) \cap (C \times C)$.

(ii) Locally bounded on $K \times L \subseteq X \times X$, if it is locally bounded at each $(x, y) \in K \times L$.

(iii) Locally bounded at $x_0 \in X$ if it is locally bounded at (x_0, x_0), i.e., there exist an open neighborhood V of x_0 and $M \in \mathbb{R}$ such that $F(x, y) \leq M$ for all x,y $\in V \cap C$.

(iv) Locally bounded on $K \subseteq X$, if it is locally bounded at each $x \in K$.

If a bifunction (not necessarily σ-monotone) $F : C \times C \to \mathbb{R}$ is locally bounded at $x_0 \in \operatorname{int} C$, then A^F is locally bounded at x_0 [2]. Consequently, if T is an operator such that G_T is locally bounded at $x_0 \in \operatorname{int} D(T)$, then T is locally bounded at x_0 since $T(x) \subseteq A^{G_T}(x)$ for all $x \in X$. This will be the main instrument for showing local boundedness of operators.

We will show that σ-monotone bifunctions are locally bounded in the interior of their domain, under mild assumptions. In case $X = \mathbb{R}^n$ we can give a constructive proof.

Proposition 3.6. *Let $X = \mathbb{R}^n$ and $C \subseteq \mathbb{R}^n$. Assume that $F : C \times C \to \mathbb{R}$ is σ-monotone and $F(x, \cdot)$ is lsc and quasiconvex for every $x \in C$. Then F is locally bounded at every point of* $\operatorname{int} C \times \operatorname{int} C$.

Proof. Let $(x_0, y_0) \in \operatorname{int} C \times \operatorname{int} C$. Since the space is finite-dimensional, we can find $z_1, z_2, \ldots, z_m \in C$ such that $V := \mathrm{co}\{z_1, z_2, \ldots, z_m\} \subseteq C$ is a neighborhood of y_0. Let $U \subseteq C$ be a compact neighborhood of x_0 in C. Set $M_k = \min_{x \in U} F(z_k, x)$; the minimum exists since $F(z_k, \cdot)$ is lsc. For every $x \in U$, $y \in V$ we find, using quasiconvexity of $F(x,\cdot)$ and σ-monotonicity of F:

$$\begin{aligned} F(x,y) &\le \max_{1\le k\le m} F(x,z_k) \\ &\le \max_{1\le k\le m} \{\sigma(z_k)\,\|x-z_k\| - F(z_k,x)\} \\ &\le \max_{1\le k\le m} \sigma(z_k) \sup_{z\in U, w\in V} \|z-w\| + \max_{1\le k\le m}(-M_k). \end{aligned}$$

Since U and V are both bounded, $\sup_{z\in U,w\in V}\|z-w\|$ is finite. We are done. □

For the general case of a Banach space X, we need the following lemma from [2], whose proof we include for the sake of completeness.

Lemma 3.7 ([2]). *Let X be a Banach space and $f : X \to \mathbb{R}\cup\{+\infty\}$ be lsc and quasiconvex. If $x_0 \in \operatorname{int}\operatorname{dom}(f)$, then f is bounded from above on a neighborhood of x_0.*

Proof. Let $\varepsilon > 0$ be such that $\overline{B}(x_0,\varepsilon) \subseteq \operatorname{dom}(f)$. Set $S_n = \{x \in \overline{B}(x_0,\varepsilon) : f(x) \le n\}$. Then S_n are convex and closed and $\bigcup_{n\in\mathbb{N}} S_n = \overline{B}(x_0,\varepsilon)$. By Baire's theorem, there exists $n \in \mathbb{N}$ such that $\operatorname{int} S_n \ne \emptyset$. Take any $x_1 \in \operatorname{int} S_n$ and any $x_2 \ne x_0$ such that $x_2 \in \overline{B}(x_0,\varepsilon)$ and $x_0 \in \operatorname{co}\{x_1,x_2\}$. Choose $n_1 > \max\{n, f(x_2)\}$. Then $x_1 \in \operatorname{int} S_{n_1}$, $x_2 \in S_{n_1}$ hence $x_0 \in \operatorname{int} S_{n_1}$ so f is bounded by n_1 at a neighborhood of x_0. □

Theorem 3.8. *Suppose X is a Banach space, C is a subset of X and $F : C\times C \to \mathbb{R}$ is a σ-monotone bifunction such that for every $x \in C$, $F(x,\cdot)$ is lsc and quasiconvex. Further, suppose that for some $x_0 \in C$ and $y_0 \in \operatorname{int} C$ there exists $\varepsilon > 0$ such that $B(y_0,\varepsilon) \subseteq C$ and for each $y \in B(y_0,\varepsilon)$, $F(y,\cdot)$ is bounded from below on $B(x_0,\varepsilon)\cap C$ (note that this bound may depend on y). Then F is locally bounded at (x_0,y_0).*

Proof. Let $\varepsilon > 0$ be as in the assumption. Define $g : B(y_0,\varepsilon) \to \mathbb{R}\cup\{+\infty\}$ by

$$g(y) := \sup\{F(x,y) : x \in B(x_0,\varepsilon)\cap C\}.$$

For every $y \in B(y_0,\varepsilon)$ and $x \in B(x_0,\varepsilon)\cap C$, σ-monotonicity of F implies

$$F(x,y) \le \min\{\sigma(x),\sigma(y)\}\|x-y\| - F(y,x) \le \sigma(y)(\varepsilon + \|y-x_0\|) - M_y$$

where M_y is a lower bound of $F(y,\cdot)$ on $B(x_0,\varepsilon)\cap C$. Therefore, g is real-valued. On the other hand, g is lsc and quasiconvex and also $y_0 \in \operatorname{int}\operatorname{dom}(g)$. By Lemma 3.7, there exists $\delta < \varepsilon$ and $M \in \mathbb{R}$ such that $g(y) \le M$ for all $y \in B(y_0,\delta)$. Then by the definition of g we get $F(x,y) \le M$ for all $y \in B(y_0,\delta)$ and $x \in B(x_0,\delta)\cap C$; i.e., F is locally bounded at (x_0,y_0). □

The condition "$F(y,\cdot)$ is bounded from below on $B(x_0,\varepsilon)\cap C$" can be easily removed by imposing some usual assumptions on the bifunction F or the space X, as shown in the following two results.

Corollary 3.9. *Suppose X is a reflexive Banach space, C is a subset of X and $F: C \times C \to \mathbb{R}$ is a σ-monotone bifunction such that for every $x \in C$, $F(x, \cdot)$ is lsc and quasiconvex. Then F is locally bounded at every point of* $\operatorname{int} C \times \operatorname{int} C$. *If in addition C is weakly closed, then F is locally bounded on $C \times \operatorname{int} C$.*

Proof. Let $x_0 \in \operatorname{int} C$. Choose $\varepsilon > 0$ such that $\overline{B}(x_0, \varepsilon) \subseteq C$. By assumption $F(x, \cdot)$ is lsc and quasiconvex, so it is weakly lsc. For every $y \in C$, $F(y, \cdot)$ attains its minimum on the weakly compact set $\overline{B}(x_0, \varepsilon)$ and so $F(y, \cdot)$ is bounded from below on $B(x_0, \varepsilon)$. Therefore, all conditions of Theorem 3.8 are satisfied. Thus F is locally bounded at every point of $\operatorname{int} C \times \operatorname{int} C$.

If in addition C is weakly closed, then for any $x_0 \in C$ and $\varepsilon > 0$, $\overline{B}(x_0, \varepsilon) \cap C$ is weakly compact and we can repeat the previous argument. □

Corollary 3.10. *Suppose X is a Banach space, C is a subset of X and $F: C \times C \to \mathbb{R}$ is a σ-monotone bifunction such that for every $x \in C$, $F(x, \cdot)$ is lsc and convex. Then F is locally bounded at any point of $C \times \operatorname{int} C$.*

Proof. Let $x_0 \in C$ and $y_0 \in \operatorname{int} C$. Choose $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subseteq C$. For every $y \in B(y_0, \varepsilon)$, the subdifferential of $\partial F(y, \cdot)$ is nonempty at y. Choose $y^* \in \partial F(y, \cdot)(y)$. Then for every $x \in B(x_0, \varepsilon) \cap C$ one has

$$F(y, x) - F(y, y) \geq \langle y^*, x - y \rangle \geq -\|y^*\| \|x - y\| \geq -\|y^*\| (\varepsilon + \|x_0 - y\|).$$

Thus $F(y, \cdot)$ is bounded from below on $B(x_0, \varepsilon) \cap C$. By Theorem 3.8, F is locally bounded at (x_0, y_0). □

We immediately obtain a generalization of Proposition 3.5 in [8] to general Banach spaces:

Corollary 3.11. *Suppose that X is a Banach space and $T: X \to 2^{X^*}$ is a pre-monotone operator. Then T is locally bounded at every point of* $\operatorname{int} D(T)$.

Proof. Apply Corollary 3.10 to G_T. □

Corollary 3.12 (Rockafellar). *Every set valued monotone operator T from X to X^* is locally bounded on* $\operatorname{int} D(T)$.

For maximal σ-monotone operators, there is a kind of converse to Corollary 3.11, generalizing the Libor Veselý theorem [11, Theorem 1.14]. We first show:

Lemma 3.13. *If T is maximal σ-monotone, then for all $x \in D(T)$ one has $T(x) + N_{D(T)}(x) \subseteq T(x)$.*

Proof. Take $w^* \in N_{D(T)}(z)$ and define

$$T_1(x) = \begin{cases} T(x) & \text{if } x \neq z, \\ T(x) + \mathbb{R}_+ w^* & \text{if } x = z. \end{cases}$$

Then $T(x) \subseteq T_1(x)$ for all $x \in D(T)$. For $z^* \in T(z)$, $y^* \in T(y)$ and $\lambda > 0$,

$$\begin{aligned}\langle z^* + \lambda w^* - y^*, z - y\rangle &= \langle z^* - y^*, z - y\rangle + \lambda\langle w^*, z - y\rangle \\ &\geq -\min\{\sigma(z), \sigma(y)\} \|z - y\|.\end{aligned}$$

Thus T_1 is σ-monotone. By the maximality of T we get $T = T_1$, which completes the proof. □

Theorem 3.14. *Suppose that T is maximal σ-monotone and σ is defined and usc on $\overline{D(T)}$. Let $x_0 \in \overline{D(T)}$. If T is locally bounded at x_0, then $x_0 \in D(T)$. If in addition $\overline{D(T)}$ is convex, then $x_0 \in \operatorname{int} D(T)$.*

Proof. Since T is locally bounded at x_0, there exists an open neighborhood U of x_0 such that $T(U)$ is bounded. Choose a sequence $\{x_n\} \subseteq D(T) \cap U$ such that $x_n \to x_0$ and choose $x_n^* \in T(x_n)$. It follows from Alaoglou's theorem that there exist a subnet $\{(x_\alpha, x_\alpha^*)\}$ of $\{(x_n, x_n^*)\}$ and $x_0^* \in X^*$ such that $x_\alpha^* \overset{w^*}{\to} x_0^*$. Since the net $\{x_\alpha^*\}$ is in the bounded set $T(U)$, we have $\langle x_\alpha^*, x_\alpha\rangle \to \langle x_0^*, x_0\rangle$. Therefore for all $(y, y^*) \in \operatorname{gr}(T)$, by upper semicontinuity of σ,

$$\begin{aligned}\langle x_0^* - y^*, x_0 - y\rangle &= \lim_\alpha \langle x_\alpha^* - y^*, x_\alpha - y\rangle \\ &\geq -\limsup_\alpha \min\{\sigma(x_\alpha), \sigma(y)\} \|x_\alpha - y\| \\ &\geq -\min\{\sigma(x_0), \sigma(y)\} \|x_0 - y\|.\end{aligned}$$

Thus (x_0, x_0^*) is σ-monotonically related with all $(y, y^*) \in \operatorname{gr}(T)$. So $x_0^* \in T(x_0)$ and $x_0 \in D(T)$.

Now let $\overline{D(T)}$ be convex. We will show that $U \subseteq \operatorname{int} \overline{D(T)}$. Indeed, if not, then U contains a boundary point of $\overline{D(T)}$. By the Bishop-Phelps theorem it will also contain a support point of $\overline{D(T)}$, i.e., there exist $z \in U \cap \overline{D(T)}$ and $0 \neq w^* \in X^*$ such that $\langle w^*, z\rangle = \sup\{\langle w^*, y\rangle : y \in \overline{D(T)}\}$. We know that T is locally bounded at z, hence $z \in D(T)$. On the other hand, $w^* \in N_{D(T)}(z)$, thus the cone $N_{D(T)}(z)$ is not equal to $\{0\}$. Then Lemma 3.13 shows that $T(z)$ cannot be bounded, a contradiction.

Thus $U \subseteq \operatorname{int} \overline{D(T)}$. Since T is locally bounded on U, we obtain $U \subseteq D(T)$, hence $x_0 \in \operatorname{int} D(T)$. □

We now deduce some properties related to local boundedness.

Proposition 3.15. *Suppose $T : X \to 2^{X^*}$ is maximal σ-monotone and σ is usc. Then*

(*i*) *The operator T is usc in $\operatorname{int} D(T)$ from the norm topology in X to the weak* topology in X^*;*

(*ii*) *If X is finite-dimensional, then for every $y \in \operatorname{int} D(T)$, $\sigma_T(y)$ is given by the following formula:*

$$\sigma_T(y) = \sup\left\{\frac{\langle x^* - y^*, y - x\rangle}{\|y - x\|} : x \neq y, (x, x^*) \in \operatorname{gr}(T), y^* \in T(y)\right\}. \qquad (8)$$

Proof. Fix $y \in \operatorname{int} D(T)$. To show upper semicontinuity at y, it is sufficient to show that for any net $\{(y_\alpha, y_\alpha^*)\}$ in $\mathrm{gr}(T)$ such that $y_\alpha \to y$ in X, there exists a weak* cluster point of $\{y_\alpha^*\}$ in $T(y)$. Since T is locally bounded at y we may assume that both $\{y_\alpha\}$ and $\{y_\alpha^*\}$ are bounded and, by selecting a subnet if necessary, $y_\alpha^* \xrightarrow{w^*} y^*$. Since $\{y_\alpha^*\}$ is bounded, we have

$$\langle y_\alpha^*, y_\alpha \rangle \to \langle y^*, y \rangle .$$

As in the proof of Proposition 2.7 we deduce that $y^* \in T(y)$.

To show part (ii), choose any sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq D(T)$ converging to y with $y \neq x_n$, and let $x_n^* \in T(x_n)$. Then the sequence $\{x_n^*\}$ is bounded. By selecting a subsequence if necessary, we may assume that x_n^* converges in norm to some $z^* \in T(y)$. Since

$$\begin{aligned} \sup \left\{ \frac{\langle x^* - y^*, y - x \rangle}{\|y - x\|} : x \neq y, (x, x^*) \in \mathrm{gr}(T), y^* \in T(y) \right\} &\geq \frac{\langle x_n^* - z^*, y - x_n \rangle}{\|y - x_n\|} \\ &\geq -\|x_n^* - z^*\| \to 0, \end{aligned}$$

relation (8) follows from relation (4). □

Next we show that under appropriate conditions, a σ-monotone bifunction is not only locally bounded, but also bounded by a small number in a neighborhood of any interior point. This is a consequence of the following more general result.

Proposition 3.16. *Suppose that $F : C \times C \to \mathbb{R}$ is a σ-monotone bifunction such that $F(x, x) = 0$ for all $x \in C$. Assume that $F(x, \cdot)$ is lsc and convex for each $x \in C$ and σ is usc. If $x_0 \in \operatorname{int} C$, then there exist an open neighborhood V of x_0 and $K \in \mathbb{R}$ such that $F(y, x) \leq K \|x - y\|$ for all $x \in V$ and $y \in C$.*

Proof. From $F(x, x) = 0$ for all $x \in C$, we infer that $A^F(x) = \partial F(x, \cdot)(x)$. Since $F(x, \cdot)$ is lsc and convex, the subdifferential of $F(x, \cdot)$ at each $x \in \operatorname{int} C$ is nonempty-valued. Thus $\operatorname{int} C \subseteq D(A^F)$, so the σ-monotone operator A^F is locally bounded at x_0. Therefore, there exist an open neighborhood $V_1 \subseteq C$ of x_0 and $K_1 \in \mathbb{R}$ such that $\|x^*\| \leq K_1$ for all $x^* \in A^F(x)$, $x \in V_1$. Since σ is usc at x_0, it is bounded from above by a number K_2 on a neighborhood V_2 of x_0. Then for each $y \in C$ and $x \in V := V_1 \cap V_2$, if we choose $x^* \in A^F(x)$ we get

$$\begin{aligned} F(y, x) &\leq -F(x, y) + \sigma(x) \|y - x\| \\ &\leq -\langle x^*, y - x \rangle + K_2 \|y - x\| \leq (K_1 + K_2) \|y - x\| . \end{aligned}$$

□

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References

[1] M. Ait Mansour, Z. Chbani, H. Riahi: Recession bifunction and solvability of noncoercive equilibrium problems, Commun. Appl. Anal. 7 (2003) 369–377.

[2] M. H. Alizadeh, N. Hadjisavvas: Local boundedness of monotone bifunctions, J. Glob. Optim., to appear, DOI 10.1007/s10898-011-9677-2.

[3] K. Aoyama, Y. Kimura, W. Takahashi: Maximal monotone operators and maximal monotone functions for equilibrium problems, J. Convex Analysis 15 (2008) 395–409.

[4] E. Blum, W. Oettli: From optimization and variational inequalities to equilibrium problems, Math. Stud. 63 (1994) 123–145.

[5] J. Borwein, S. Fitzpatrick, R. Girgensohn: Subdifferentials whose graphs are not norm $\times$weak* closed, Canad. Math. Bull. 46 (2003) 538–545.

[6] N. Hadjisavvas, H. Khatibzadeh: Maximal monotonicity of bifunctions, Optimization 59 (2010) 147-160.

[7] N. Hadjisavvas, S. Komlósi, S. Schaible: Handbook of Generalized Convexity and Generalized Monotonicity, Springer, New York (2005).

[8] A. N. Iusem, G. Kassay, W. Sosa: An existence result for equilibrium problems with some surjectivity consequences, J. Convex Analysis 16 (2009) 807–826.

[9] A. Jofré, D. T. Luc, M. Théra: ε-subdifferential and ε-monotonicity, Nonlinear Anal. 33 (1998) 71–90.

[10] D. T. Luc, H. V. Ngai, M. Théra: On ε-monotonicity and ε-convexity, in: A. Ioffe et al. (eds.), Calculus of Variations and Differential Equations (Haifa, 1998), Res. Notes Math. Ser. 410, Chapman & Hall, Boca Raton (1999) 82–100.

[11] R. Phelps: Lectures on maximal monotone operators, Extr. Math. 12 (1997) 193–230.