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Well-Posedness for Mixed Quasivariational-Like Inequalities

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Abstract In this paper, we introduce concepts of well-posedness, and well-posedness in the generalized sense, for mixed quasivariational-like inequalities where the underlying map is multivalued. We give necessary and sufficient conditions for the various kinds of well-posedness to occur. Our results generalize and strengthen previously found results for variational and quasivariational inequalities.

Keywords Mixed quasivariational-like inequalities · Well-posedness · Well-posedness in the generalized sense · Multivalued maps · Measure of noncompactness

1 Introduction

Let E be a real Banach space, E^* its dual, and K be a nonempty closed convex subset of E. Let further $S : K \to 2^K$ and $F : K \to 2^{E^*}$ be multivalued maps with nonempty

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values, $\eta : K \times K \to E$ be a single-valued mapping, and $f : K \to \mathbb{R}$ a real-valued function. We consider the mixed quasivariational-like inequality problem, which is to find $x_o \in K$ such that, for some $u_o \in F(x_o)$,

$$\text{(MQVLI)} \quad x_o \in S(x_o) \quad \text{and} \quad \langle u_o, \eta(x_o, y)\rangle + f(x_o) - f(y) \le 0, \quad \forall y \in S(x_o).$$

If $f(x) = 0$, $\forall x \in K$, $\eta(x, y) = x - y$, $\forall(x, y) \in K \times K$, and F is a single-valued mapping, then (MQVLI) reduces to the classical quasivariational inequality problem (Refs. [2, 4, 15]), which consists in finding a point $x_o \in K$ such that

$$\text{(QVI)} \quad x_o \in S(x_o) \quad \text{and} \quad \langle F(x_o), x_o - y\rangle \le 0, \quad \forall y \in S(x_o).$$

Quasivariational inequalities were extensively studied in recent years due to their applicability to many problems in economics and engineering (Ref. [7]).

When E is a real Hilbert space, F is a single-valued mapping, and $S(x) = K$, $\forall x \in K$, (MQVLI) reduces to the mixed variational-like inequality problem, which consists in finding a point $x_o \in K$ such that

$$\text{(MVLI)} \quad \langle F(x_o), \eta(x_o, y)\rangle + f(x_o) - f(y) \le 0, \quad \forall y \in K.$$

This problem has been studied intensively (see, e.g., Refs. [1, 21] and the references therein). Recently, Ansari and Yao (Ref. [1]) and Zeng (Ref. [21]) developed some iterative schemes for finding the approximate solutions of (MVLI) and proved that these approximate solutions strongly converge to the exact solution of (MVLI).

If $S(x) = K$, $\forall x \in K$, $\eta(x, y) = x - y$, $\forall(x, y) \in K \times K$, and F is of the form $F = A \circ G$ where $G : K \to 2^{E^*}\backslash\{\emptyset\}$ is a multivalued map and $A : E^* \to E^*$ is single-valued, then (MQVLI) reduces to the generalized mixed variational inequality problem, which consists in finding a point $x_o \in K$ such that, for some $u_o \in G(x_o)$,

$$\text{(GMVI)} \quad \langle Au_o, x_o - y\rangle + f(x_o) - f(y) \le 0, \quad \forall y \in K.$$

This problem has been studied intensively. Recently, Zeng and Yao (Ref. [23]), Schaible, Yao and Zeng (Ref. [17]), and Zeng (Ref. [22]) considered the existence of solutions of generalized mixed variational inequalities (GMVI), and proposed and analyzed iterative algorithms for finding approximate solutions of (GMVI).

Very recently, Lignola (Ref. [10]) introduced and investigated the concepts of well-posedness and L-well-posedness for quasivariational inequalities (with a single-valued map F) having a unique solution and the concepts of well-posedness and L-well-posedness in the generalized sense for quasivariational inequalities having more than one solution, in analogy to the corresponding concepts for variational inequalities (Refs. [6, 12]). In this paper, inspired by these concepts of well-posedness for (QVI), we introduce and study the concepts of well-posedness and L-well-posedness for mixed quasivariational-like inequalities having a unique solution and the concepts of well-posedness and L-well-posedness in the generalized sense for mixed quasivariational-like inequalities having more than one solution. The results obtained in this paper generalize the results of Ref. [10] in the more general setting of (MQVLI), but also strengthen some of them, by imposing weaker assumptions on the multivalued maps S and F. A necessary and sufficient condition for well-posedness is formulated in terms of the diameters of the approximate solution sets. In a similar way, well-posedness in the generalized sense is shown to be equivalent to a condition involving a regular measure of noncompactness of the approximate solution sets.

This paper is organized as follows: in Sect. 2, we introduce the necessary notation and definitions, and we show two basic lemmas, one concerning a generalization of the so-called Minty Lemma for variational inequalities, and another on a property of the Painlevé-Kuratowski limit inferior of a sequence of convex sets in a finite-dimensional space. In Sect. 3 we establish necessary and sufficient conditions for well-posedness and L-well-posedness of (MQVLI), while in Sect. 4 we do the same for well-posedness and L-well-posedness in the generalized sense.

2 Preliminaries

Given a Banach space E, for any $x \in E$ and $A \subseteq E$ we will set $d(x, A) = \inf\{\|x - y\| : y \in A\}$ and $\operatorname{diam} A = \sup\{\|y - z\| : y, z \in A\}$ for the distance of x from A and the diameter of A, respectively. We will denote by $\overline{A}$, $\operatorname{int} A$ and $\operatorname{ri} A$ the closure, the interior and the relative interior of A, respectively.

We recall the notion of Mosco convergence (Ref. [14]). A sequence $(S_n)_n$ of subsets of E Mosco converges to a set S if

$$S = w - \limsup_n S_n = \liminf_n S_n,$$

where $w - \limsup_n S_n$ and $\liminf_n S_n$ are, respectively, the Painlevé-Kuratowski weak limit superior, and the Painlevé-Kuratowski strong limit inferior of $(S_n)_n$, i.e.,

$$w - \limsup_n S_n = \{x \in E : \exists n_k \uparrow +\infty, n_k \in \mathbb{N}, \exists x_{n_k} \in S_n, \text{ with } x_{n_k} \rightharpoonup x\},$$

$$\liminf_n S_n = \{x \in E : \exists x_n \in S_n, n \in \mathbb{N}, \text{ with } x_n \to x\}.$$

We shall use the usual abbreviations usc and lsc for “upper semicontinuous” and “lower semicontinuous”, respectively. For any $x, y \in E$, $[x, y]$ will denote the line segment $\{tx + (1 - t)y : t \in [0, 1]\}$, while $[x, y)$ and (x, y) are defined analogously. We will frequently use s, w and w^* to denote, respectively, the norm topology on E, the weak topology on E and the weak* topology on E^*. Given a convex set K, a multivalued map $F : K \to 2^{E^*}$ will be called upper hemicontinuous if its restriction on any line segment $[x, y] \subseteq K$ is usc with respect to the w^* topology on E^*. We refer the reader to Ref. [8] for basic facts about multivalued maps.

Definition 2.1 Let E be a real Banach space and K be a nonempty subset of E. Let $\eta : K \times K \to E$ be a map, $F : K \to 2^{E^*}$ a nonempty-valued multifunction, and $f : K \to \mathbb{R}$ a real-valued function. Then, F is said to be:

(i) η-pseudomonotone with respect to f if, for any $x, y \in K$, $u \in F(x)$ and $v \in F(y)$,

$$\langle u, \eta \rangle + f(x) - f(y) \le 0 \quad \Rightarrow \quad \langle v, \eta(x, y)\rangle + f(x) - f(y) \le 0.$$

When $f = 0$, we will simply say that F is η-pseudomonotone.

(ii) η-monotone if, for any $x, y \in K$, $u \in F(x)$ and $v \in F(y)$,

$$\langle u - v, \eta(x, y)\rangle \ge 0.$$

It is obvious that an η-monotone map is η-pseudomonotone with respect to any f.

The following is a kind of "Minty's Lemma" showing that (MQVLI) can be stated in an equivalent form.

Lemma 2.1 *Let E be a real Banach space and let S_o be a nonempty, closed and convex subset of E. Let further $f : S_o \to \mathbb{R}$ be a convex function, $\eta : S_o \times S_o \to E$ a mapping satisfying $\eta(x, x) = 0$ for all $x \in S_o$, and let $F : S_o \to 2^{E^*}$ be a multifunction with nonempty, weakly* compact values which is η-pseudomonotone with respect to f and upper hemicontinuous. If the map $y \mapsto \langle u, \eta(x, y)\rangle$ is concave for each $(u, x) \in F(S_o) \times S_o$, then for each $x_o \in S_o$, the following statements are equivalent*:

(i) $\forall y \in S_o,\ \exists v \in F(x_o)$, *such that* $\langle v, \eta(x_o, y)\rangle + f(x_o) - f(y) \le 0$;
(ii) $\forall y \in S_o,\ \forall v \in F(y)$, $\langle v, \eta(x_o, y)\rangle + f(x_o) - f(y) \le 0$.

We will not prove the above lemma since the argument for its proof is actually contained in the proof of the next one. The interest of the next lemma is that a condition stronger that (i) in Lemma 2.1 in which v is actually independent of $y \in S_o$, is shown to be equivalent to a condition weaker that (ii), in which the same inequality is imposed to hold for y in a smaller set S_1, and for some (not all) $v \in F(y)$.

Lemma 2.2 *Let E be a real Banach space and let S_o be a nonempty, closed and convex subset of E. Let further $f : S_o \to \mathbb{R}$ be a convex lsc function, $\eta : S_o \times S_o \to E$ a mapping satisfying $\eta(x, x) = 0$ for all $x \in S_o$, and $F : S_o \to 2^{E^*}$ a multifunction with nonempty, weakly* compact convex values which is η-pseudomonotone with respect to f and upper hemicontinuous. Assume that the map $y \mapsto \langle u, \eta(x, y)\rangle$ is concave and usc for each $(u, x) \in F(S_o) \times S_o$. If S_1 is a convex subset of S_o with the property that, for each $x \in S_o$ and each $y \in S_1$, $(x, y] \subseteq S_1$, then for each $x_o \in S_o$, the following statements are equivalent*:

(i′) $\exists v_o \in F(x_o)$ *such that* $\forall y \in S_o,\ \langle v_o, \eta(x_o, y)\rangle + f(x_o) - f(y) \le 0$;
(ii′) $\forall y \in S_1,\ \exists v \in F(y)$ *such that* $\langle v, \eta(x_o, y)\rangle + f(x_o) - f(y) \le 0$.

Proof Implication (i′)⇒(ii′) is an obvious consequence of the η-pseudomonotonicity of F with respect to f.

Suppose that (ii′) holds. Given any $y \in S_1$, define $y_n = \frac{1}{n}y + (1 - \frac{1}{n})x_o$ for $n \in \mathbb{N}$. By our assumption on S_1, $y_n \in S_1$ for all $n \in \mathbb{N}$. According to condition (ii′), there exists $v_n \in F(y_n)$ such that $\langle v_n, \eta(x_o, y_n)\rangle + f(x_o) - f(y_n) \le 0$. Then,

$$0 \ge \langle v_n, \eta(x_o, y_n)\rangle + f(x_o) - f(y_n)$$

$$= \left\langle v_n, \eta\left(x_o, \frac{1}{n}y + \left(1 - \frac{1}{n}\right)x_o\right)\right\rangle + f(x_o) - f\left(\frac{1}{n}y + \left(1 - \frac{1}{n}\right)x_o\right)$$

$$\geq \frac{1}{n}\langle v_n, \eta(x_o, y)\rangle + \left(1 - \frac{1}{n}\right)\langle v_n, \eta(x_o, x_o)\rangle + f(x_o) - \frac{1}{n}f(y) - \left(1 - \frac{1}{n}\right)f(x_o)$$

$$= \frac{1}{n}[\langle v_n, \eta(x_o, y)\rangle + f(x_o) - f(y)],$$

which implies that

$$\langle v_n, \eta(x_o, y)\rangle + f(x_o) - f(y) \leq 0, \quad \text{for some } v_n \in F(y_n),\ n \in \mathbb{N}. \tag{1}$$

Since F is a weakly* compact valued multifunction which is (s, w^*)-usc on the line segment $[x_o, y] = \{ty + (1-t)x_o, t \in [0,1]\}$, the image $F([x_o, y])$ is weakly* compact; hence $\{v_n\}_n$ has a subnet weakly* converging to some $v \in E^*$. In addition, the restriction of F on $[x_o, y]$ is closed, hence $v_n \in F(y_n)$ and $\lim_n y_n = x_o$ imply $v \in F(x_o)$. By taking the limit of the subnet in (1), we obtain

$$\forall y \in S_1,\ \exists v \in F(x_o) : \langle v, \eta(x_o, y)\rangle + f(x_o) - f(y) \leq 0. \tag{2}$$

Define the bifunction $\phi(v, y)$ on $F(x_o) \times S_o$ by

$$\phi(v, y) = \langle v, \eta(x_o, y)\rangle + f(x_o) - f(y). \tag{3}$$

According to our assumptions, for each $y \in S_o$, $\phi(\cdot, y)$ is weakly* lsc and quasiconvex on the weakly* compact convex set $F(x_o)$, while for each $v \in F(x_o)$, $\phi(v, \cdot)$ is usc and quasiconcave on the convex set S_1. Hence, according to the Sion minimax Theorem (Ref. [18]),

$$\sup_{y \in S_1} \min_{v \in F(x_o)} \phi(v, y) = \min_{v \in F(x_o)} \sup_{y \in S_1} \phi(v, y).$$

By (2), we have $\sup_{y \in S_1} \min_{v \in F(x_o)} \phi(v, y) \leq 0$; hence, $\min_{v \in F(x_o)} \sup_{y \in S_1} \phi(v, y) \leq 0$, which implies that there exists $v_o \in F(x_o)$ such that

$$\langle v_o, \eta(x_o, y)\rangle + f(x_o) - f(y) \leq 0, \tag{4}$$

for each $y \in S_1$.

Finally, for each $y \in S_o$ choose $z \in S_1$ and a sequence $\{y_n\}_n$ in $(y, z] \subset S_1$ converging to y. The function $\phi(v, \cdot)$ is usc and concave on S_o; hence, its restriction on any line segment is continuous (Ref. [16]). Accordingly, (4) implies

$$\langle v_o, \eta(x_o, y)\rangle + f(x_o) - f(y) = \lim_n (\langle v_o, \eta(x_o, y_n)\rangle + f(x_o) - f(y_n)) \leq 0.$$

Hence (ii′) holds. □

Remark 2.1 Note that the assumption that F is η-pseudomonotone with respect to f is used only in the implication (i)⇒(ii).

Convex subsets S_1 of a convex set S_o with the property $(x, y] \subseteq S_1$ for each $x \in S_o$ and $y \in S_1$ are: the set S_o itself, the interior $\operatorname{int} S_o$ (if nonempty), and various kinds of generalized interiors (if nonempty) such as the set of inner points $\operatorname{inn} S_o$ (Ref. [20]), the relative interior $\operatorname{ri} S_o$, etc.

It should be noted also that if the map F is onto (i.e., $F(S_o) = E^*$) then the condition "the map $y \mapsto \langle u, \eta(x, y)\rangle$ is concave and usc for each $(u, x) \in F(S_o) \times S_o$" is equivalent to the condition "the map $y \to \eta(x, y)$ is affine and (s, w)-continuous for each $x \in E$". Indeed, since $\langle u, \eta(x, \cdot)\rangle$ and $\langle -u, \eta(x, \cdot)\rangle$ are both usc, we deduce that $\langle u, \eta(x, \cdot)\rangle$ is continuous for each $u \in E^*$, i.e., $\eta(x, \cdot)$ is (s, w)-continuous. A similar argument shows that $\eta(x, \cdot)$ is affine, i.e., convex and concave.

Recall that, after Tykhonov (Ref. [19]) first defined well-posedness (resp., well-posedness in the generalized sense) of a minimization problem, some well-posedness notions for variational inequalities were introduced in Ref. [12] using the Auslender gap function and in Ref. [5] using the Fukushima gap function. Recently, inspired by the well-posedness notions for variational inequalities, Lignola (Ref. [10]) defined some concepts of well-posedness for quasivariational inequalities. In this paper, following the same approach as in Ref. [10], we introduce the concepts of well-posedness for mixed quasivariational-like inequalities with the aid of suitable approximating sequences.

Definition 2.2 A sequence $\{x_n\}_n$ in K is said to be an approximating sequence [resp., an L-approximating sequence] for the mixed quasivariational-like inequality (MQVLI) if there exists a sequence $\{u_n\}_n$ in E^* with $u_n \in F(x_n)$, $\forall n \in \mathbb{N}$, and a sequence $\{\varepsilon_n\}_n$ in $\mathbb{R}$ with $\varepsilon_n \downarrow 0$, such that

$$d(x_n, S(x_n)) \le \varepsilon_n \quad \text{and} \quad \langle u_n, \eta(x_n, y)\rangle + f(x_n) - f(y) \le \varepsilon_n, \quad \forall y \in S(x_n),\ n \in \mathbb{N}.$$

[resp., if there exists a sequence $\{\varepsilon_n\}_n$ in $\mathbb{R}$ with $\varepsilon_n \downarrow 0$ and such that

$$\begin{aligned} &d(x_n, S(x_n)) \le \varepsilon_n \quad \text{and} \quad \langle v, \eta(x_n, y)\rangle + f(x_n) - f(y) \le \varepsilon_n, \\ &\forall y \in S(x_n),\ v \in F(y),\ n \in \mathbb{N}.] \end{aligned}$$

If $f(x) = 0$, $\forall x \in K$, $\eta(x, y) = x - y$, $\forall (x, y) \in K \times K$ and F is single-valued, then the mixed quasivariational-like inequality reduces to the quasivariational inequality considered by Lignola (Ref. [10]) and our definition of approximating sequences reduces to the definition introduced in Ref. [10].

Definition 2.3 A mixed quasivariational-like inequality is termed well-posed [resp., L-well-posed] if it has a unique solution x_o and every approximating [resp., L-approximating] sequence $\{x_n\}_n$ strongly converges to x_o.

In order to characterize the well-posedness of the quasivariational inequality, Lignola (Ref. [10]) defined some concepts of approximate solutions for quasivariational inequalities. Motivated by these concepts, for every positive number ε, we consider the sets

$$Q_\varepsilon = \{x \in K : d(x, S(x)) < \varepsilon \text{ and } \\ \exists u \in F(x) : \langle u, \eta(x, y)\rangle + f(x) - f(y) \le \varepsilon, \forall y \in S(x)\}$$

and

$$L_\varepsilon = \{x \in K : d(x, S(x)) < \varepsilon \text{ and } \\ \langle v, \eta(x, y)\rangle + f(x) - f(y) \le \varepsilon,\ \forall y \in S(x),\ v \in F(y)\}.$$

Remark 2.2 (a) Note that any solution of (MQVLI) belongs to Q_ε for all $\varepsilon > 0$. Also, the maps $\varepsilon \to Q_\varepsilon$ and $\varepsilon \to L_\varepsilon$ are increasing (i.e., $\varepsilon \le \varepsilon'$ implies $Q_\varepsilon \subseteq Q_{\varepsilon'}$ and $L_\varepsilon \subseteq L_{\varepsilon'}$). Finally, if F is η-monotone, then obviously $Q_\varepsilon \subseteq L_\varepsilon$ for all $\varepsilon > 0$. If F is only η-pseudomonotone with respect to f the inclusion $Q_\varepsilon \subseteq L_\varepsilon$ does not necessarily hold; however, it is easy to see that in this case again, every solution of (MQVLI) belongs to L_ε for all $\varepsilon > 0$.

(b) It should also be noted that, under some mild assumptions (namely, that the values of F are weak* compact and convex, S is convex-valued, f is convex and lsc, and $\langle v, \eta(x, \cdot)\rangle$ is concave and usc for each $(v, x) \in F(K) \times K$), by applying Sion's minimax theorem to the bifunction

$$\phi(u, y) = \langle u, \eta(x, y)\rangle + f(x) - f(y), \quad (u, y) \in F(x) \times S(x),$$

we deduce that Q_ε can be equivalently defined as

$$Q_\varepsilon = \{x \in K : d(x, S(x)) < \varepsilon \text{ and } \\ \forall y \in S(x),\ \exists u \in F(x) : \langle u, \eta(x, y)\rangle + f(x) - f(y) \le \varepsilon\},$$

where u might depend on y. This holds also for the definition of approximating sequence.

The following lemma, taken from Ref. [11], will be useful in the next sections.

Lemma 2.3 *Let $\{H_n\}_n$ be a sequence of nonempty subsets of the space E such that*:

(i) *H_n is convex for every $n \in \mathbb{N}$*;
(ii) $H_o \subseteq \liminf_n H_n$;
(iii) *there exists $m \in N$ such that* $\operatorname{int} \bigcap_{n \ge m} H_n \ne \emptyset$.

Then, for every $x_o \in \operatorname{int} H_o$, there exists a positive number δ such that

$$B(x_0, \delta) \subseteq H_n, \quad \forall n \ge m.$$

If E is a finite-dimensional space, then Assumption (iii) *can be replaced by*

(iii′) $\operatorname{int} H_o \ne \emptyset$.

According to the above lemma, for convex sets in a finite-dimensional space, if $x_o \in \operatorname{int} H_o \subseteq \liminf_n H_n$ then $x_o \in H_n$ for all n sufficiently large. The usefulness of the lemma is of course restricted to the case where H_o is full-dimensional, i.e., generates H_o. In case H_o generates a hyperplane of E then, as the following proposition shows, x_o is the limit of a sequence $x_n \in H_n$, $n \in \mathbb{N}$, lying on a straight line.

Proposition 2.1 *Let H_o and H_n, $n \in \mathbb{N}$, be nonempty convex subsets in $\mathbb{R}^k$ such that $H_o \subseteq \liminf_n H_n$ and $\dim H_o = k-1$. Then, for each $x \in \operatorname{ri} H_o$ and each $v \neq 0$ not belonging to the affine subspace generated by H_o, there exists a sequence $(x_n)_n$ in the line $\{x + t(v-x), t \in \mathbb{R}\}$, such that $\lim_n x_n = x_o$ and $x_n \in H_n$ for all $n \in \mathbb{N}$.*

Proof By translating the sets H_o, H_n and the points x, v by $-x$, we may assume that $x = 0$. Then the affine subspace X generated by H_o is in fact a subspace. Since $\dim X = \dim H_o = k-1$ and $0 \in \operatorname{ri} H_o$, we can find $k-1$ linearly independent vectors $v^i, i = 1, \ldots, k-1$ in H_o. By choosing these vectors sufficiently small we will also have $v_k := -(v_1 + \cdots + v_{k-1}) \in H_o$. Let $\varepsilon > 0$ be small enough so that for each $y^i \in B(v^i, \varepsilon)$, the set $\{y^1, \ldots, y^{k-1}\}$ is linearly independent (this is possible since linear independence can be characterized through determinants). Then each $y^k \in B(v^k, \varepsilon)$ can be written in a unique way as $y^k = \sum_{i=1}^{k-1} \lambda^i y^i$. By continuity of the coefficients we have that for y^k sufficiently close to v^k, λ^i is close to -1 for all $i = 1, \ldots, k-1$ and in particular is negative.

For each $i = 1, \ldots, k$ there exists a sequence $\{z_n^i\}_n$ converging to v^i such that $z_n^i \in H_n$. The vector v together with X generates $\mathbb{R}^k$, thus each z_n^i can be uniquely written as $z_n^i = y_n^i + t_n^i v$ where $y_n^i \in X$ is the oblique projection of z_n^i onto X along v. Since $\lim_n z_n^i = v^i$ and the projection is continuous, we have $\lim_n y_n^i = v^i$ and $\lim_n t_n^i = 0$. For sufficiently large n, $y_n^i \in B(v^i, \varepsilon)$. By the discussion above, y_n^k can be written as $y_n^k = -\sum_{i=1}^{k-1} \lambda_n^i y_n^i$ where λ_n^i are positive. It follows that there exist positive numbers $\mu_n^i, n \in \mathbb{N}, i = 1, \ldots, k$, such that $\sum_{i=1}^k \mu_n^i = 1$ and $\sum_{i=1}^k \mu_n^i y_n^i = 0, n \in \mathbb{N}$.

Set $x_n = \sum_{i=1}^k \mu_n^i z_n^i$, $n \in \mathbb{N}$. Then, $x_n \in H_n$ by convexity and

$$x_n = \sum_{i=1}^k \mu_n^i \left(y_n^i + t_n^i v \right) = v \sum_{i=1}^k \mu_n^i t_n^i,$$

with

$$\lim_n x_n = v \lim_n \sum_{i=1}^k \mu_n^i t_n^i = 0.$$

The proof is complete. □

3 Case of a Unique Solution

We now establish some necessary and sufficient conditions for well-posedness and L-well-posedness for mixed quasivariational-like inequalities.

Theorem 3.1 *Let E be a real Banach space and let K be a nonempty, closed and convex subset of E. Let $f : K \to \mathbb{R}$ be convex and lsc and let $\eta : K \times K \to E$ be a mapping with $\eta(x, x) = 0$, $\forall x \in K$, which is (s, w)-continuous in each of its variables separately. Let $S : K \to 2^K$ and $F : K \to 2^{E^*}$ be multifunctions. Assume that the following conditions hold*:

(i) *the multifunction S has nonempty convex values and, for each sequence $(x_n)_n$ in K converging to x_0, the sequence $(S(x_n))_n$ Mosco converges to $S(x_0)$*;
(ii) *for every converging sequence $\{w_n\}_n$, there exists $m \in \mathbb{N}$ such that*

$$\operatorname{int} \bigcap_{n \ge m} S(w_n) \neq \emptyset;$$

(iii) *the multifunction F has nonempty, weakly* compact convex values and is upper hemicontinuous and η-monotone*;
(iv) *the functional $y \mapsto \langle u, \eta(x, y)\rangle$ is concave for each $(u, x) \in F(K) \times K$.*

Then, (MQVLI) *is well-posed if and only if*

$$Q_\varepsilon \neq \emptyset, \quad \forall \varepsilon > 0, \quad \textit{and} \quad \lim_{\varepsilon \to 0} \operatorname{diam} Q_\varepsilon = 0. \tag{5}$$

The proof of the theorem relies on the following lemma, which will be also used in the next section.

Lemma 3.1 *Let the same assumptions be made as in Theorem 3.1. Let $\{x_n\}_n \subseteq K$ be an approximating sequence. If $\{x_n\}_n$ converges to some $x_o \in K$, then x_o is a solution of* (MQVLI).

Proof Since $\{x_n\}_n$ is an approximating sequence, there exists a sequence $\{u_n\}_n$ in E^* with $u_n \in F(x_n)$, $\forall n \in \mathbb{N}$, and a sequence $\{\varepsilon_n\}_n$ in $\mathbb{R}$ with $\varepsilon_n \downarrow 0$ such that $d(x_n, S(x_n)) < \varepsilon_n$ and

$$\langle u_n, \eta(x_n, y)\rangle + f(x_n) - f(y) \le \varepsilon_n, \quad \forall y \in S(x_n),\ n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, choose $x_n' \in S(x_n)$ such that $\|x_n - x_n'\| < 2d(x_n, S(x_n)) < 2\varepsilon_n$. Then obviously $x_n' \to x_0$. By Mosco convergence we have $\liminf_n S(x_n) = S(x_0)$. Thus, $x_0 \in S(x_0)$.

Assumption (ii) applied to the constant sequence $w_n = x_0$, $n \in \mathbb{N}$, implies that $\operatorname{int} S(x_o) \neq \emptyset$. Choose $y \in \operatorname{int} S(x_o)$. Since $S(x_o) = \liminf_n S(x_n)$, Lemma 2.3 implies that $y \in S(x_n)$ for n sufficiently large. Using successively the fact that $\eta(\cdot, y)$ is (s, w)-continuous, f is lsc, F is η-monotone, and $\{x_n\}_n$ is an approximating sequence, we obtain, for every $v \in F(y)$,

$$\begin{aligned} \langle v, \eta(x_o, y)\rangle + f(x_o) - f(y) &\le \liminf_n \{\langle v, \eta(x_n, y)\rangle + f(x_n) - f(y)\} \\ &\le \liminf_n \{\langle u_n, \eta(x_n, y)\rangle + f(x_n) - f(y)\} \\ &\le \lim_n \varepsilon_n = 0. \end{aligned}$$

Thus, for every $y \in \operatorname{int} S(x_o)$ and every $v \in F(y)$, $\langle v, \eta(x_o, y)\rangle + f(x_o) - f(y) \le 0$ holds. Applying Lemma 2.2 for $S_o = S(x_o)$ and $S_1 = \operatorname{int} S(x_o)$ we obtain the existence of $v_o \in F(x_o)$ such that $\forall y \in S(x_o)$, $\langle v_o, \eta(x_o, y)\rangle + f(x_o) - f(y) \le 0$. Thus, x_o is a solution of (MQVLI). □

Proof of Theorem 3.1 Suppose that (5) holds. Let us show that there exists at most one solution of (MQVLI). Indeed, if there existed two solutions z_1 and z_2, then we would have $z_1, z_2 \in Q_\varepsilon$ for all $\varepsilon > 0$, thus $\lim_{\varepsilon\to 0} \operatorname{diam} Q_\varepsilon \geq \|z_1 - z_2\|$, a contradiction. Note also that there exist approximate sequences for (MQVLI); indeed, for any sequence $\{\varepsilon_n\}_n$ in $\mathbb{R}_n$ with $\varepsilon_n \downarrow 0$, and any choice of $x_n \in Q_{\varepsilon_n}$ (which is nonempty by assumption), $\{x_n\}_n$ is an approximate sequence. Hence, it suffices to show that any approximate sequence converges to a solution of (MQVLI).

Let $\{x_n\}_n \subseteq K$ be an approximating sequence for (MQVLI). Then $x_n \in Q_{\varepsilon_n} \subseteq Q_{\varepsilon_m}$ for each $n \geq m$. Using (5) we deduce that $\{x_n\}$ is a Cauchy sequence, thus converges to some $x_0 \in K$. Applying Lemma 3.1 we deduce that x_o is a solution of (MQVLI) and the problem is well-posed.

The converse is standard: assume that the problem (MQVLI) is well-posed. Then $Q_\varepsilon \neq \emptyset$ for every $\varepsilon > 0$, since Q_ε contains the unique solution of (MQVLI). Given any sequence $\{\varepsilon_n\}_n$ such that $\varepsilon_n \to 0$, for each $n \in \mathbb{N}$ we can find $x_n^{(1)}, x_n^{(2)} \in Q_{\varepsilon_n}$ such that $\|x_n^{(1)} - x_n^{(2)}\| \geq \frac{1}{2} \operatorname{diam} Q_{\varepsilon_n}$. The sequences $\{x_n^{(1)}\}$ and $\{x_n^{(2)}\}$ are approximating. Since the problem is well-posed, both sequences converge to the unique solution. Thus, $\lim_n \|x_n^{(1)} - x_n^{(2)}\| = 0$. This implies that $\lim_n \operatorname{diam} Q_{\varepsilon_n} = 0$ and, since this is true for any sequence $\{\varepsilon_n\}_n$, we deduce that $\lim_{\varepsilon\to 0} \operatorname{diam} Q_\varepsilon = 0$. The proof is complete. □

Remark 3.1 (a) It follows from the above proof and Lemma 2.3 that in case E is finite-dimensional, we can replace condition (ii) by the assumption that $\operatorname{int} S(x) \neq \emptyset$ for all $x \in K$.

(b) In some papers (Refs. [10, 13]) it is assumed that the set-valued mapping S has nonempty convex values, and is (s,s) lsc, (s,w) closed (with E reflexive) and (s,w) subcontinuous on K. However, it is known that a map is (s,w) closed and (s,w) subcontinuous if and only if it is (s,w) usc and has weakly compact values (see for instance Ref. [9]). Thus our assumption (i) is weaker, and it avoids weak compactness of the values so that $S(x)$ can be unbounded.

If we strengthen the continuity requirements, we may avoid condition (ii) and also monotonicity of F.

Theorem 3.2 *Let E be a real Banach space with the dual E^* and let K be a nonempty, closed and convex subset of E. Let $f : K \to \mathbb{R}$ be convex and continuous, and let $\eta : K \times K \to E$ be a mapping with $\eta(x,x) = 0$, $\forall x \in K$, which is $(s \times s, s)$-continuous. Let $S : K \to 2^K$ and $F : K \to 2^{E^*}$ be multifunctions. Assume that the following conditions hold*:

(i) *the multifunction S has nonempty convex values and, for each sequence $(x_n)_n$ in K converging to x_0, the sequence $\{S(x_n)\}_n$ Mosco converges to $S(x_0)$*;
(ii) *the multifunction F has nonempty, weakly* compact and convex values, and is (s, w^*)-usc*;
(iii) *the functional $y \mapsto \langle u, \eta(x,y)\rangle$ is concave for each $(u,x) \in F(K) \times K$.*

Then, (MQVLI) *is well-posed if and only if*

$$Q_\varepsilon \neq \emptyset, \quad \forall \varepsilon > 0, \quad \textit{and} \quad \lim_{\varepsilon\to 0} \operatorname{diam} Q_\varepsilon = 0.$$

We separate again the main argument of the proof, in view of subsequent use.

Lemma 3.2 *Let the assumptions be as in Theorem 3.2. Let $\{x_n\}_n \subseteq K$ be an approximating sequence. If $\{x_n\}_n$ converges to some $x_o \in K$, then x_o is a solution of* (MQVLI).

Proof Since $\{x_n\}_n$ an approximating sequence, there exists a sequence $\{u_n\}_n$ in E^* with $u_n \in F(x_n)$, $\forall n \in \mathbb{N}$, and there exists a sequence $\{\varepsilon_n\}_n$ in $\mathbb{R}$ with $\varepsilon_n \downarrow 0$ such that $d(x_n, S(x_n)) < \varepsilon_n$ and

$$\langle u_n, \eta(x_n, y)\rangle + f(x_n) - f(y) \le \varepsilon_n, \quad \forall y \in S(x_n),\ n \in \mathbb{N}.$$

As in Lemma 3.1, we infer that $x_o \in S(x_o)$. Since $S(x_n)$ Mosco converges to $S(x_o)$, for every $y \in S(x_o)$ there exists a sequence $y_n \in S(x_n)$, $n \in \mathbb{N}$, such that $\lim y_n = y$ in the strong topology. By our assumption on η, the sequence $\{\eta(x_n, y_n)\}_n$ converges strongly to $\eta(x_o, y)$. Since F is (s, w^*) usc with weakly* compact values, the image of the sequence $\{x_n\}_n$ through F is relatively weakly* compact; hence there exists a subnet $\{u_\alpha\}_\alpha$ of $\{u_n\}_n$ weakly* converging to some $u_o \in F(x_o)$. The set $\bigcup_n F(x_n)$ is bounded by some number $m > 0$. Consequently, the net $\{u_\alpha\}_\alpha$ is also bounded by m. It follows that

$$\begin{aligned}
&|\langle u_\alpha, \eta(x_a, y_\alpha)\rangle - \langle u_o, \eta(x_o, y)\rangle| \\
&\quad \le |\langle u_\alpha, \eta(x_a, y_\alpha) - \eta(x_o, y)\rangle| + |\langle u_o - u_\alpha, \eta(x_o, y)\rangle| \\
&\quad \le m\, |\eta(x_a, y_\alpha) - \eta(x_o, y)| + |\langle u_\alpha - u_o, \eta(x_o, y)\rangle| \to 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\langle u_o, \eta(x_o, y)\rangle + f(x_o) - f(y) \\
&\quad = \lim_\alpha (\langle u_\alpha, \eta(x_a, y_\alpha)\rangle + f(x_\alpha) - f(y_\alpha)) \le \lim_\alpha \varepsilon_\alpha = 0.
\end{aligned}$$

Thus, x_o is a solution of (MQVLI). □

Proof of Theorem 3.2 Assume that condition (5) holds. As in the proof of Theorem 3.1, we have to show that if $\{x_n\}_n \subseteq K$ is an approximating sequence for (MQVLI), then it converges to a solution of (MQVLI). As in Theorem 3.1, $\{x_n\}_n$ converges to a point $x_o \in K$. Applying Lemma 3.2 we deduce that x_o is a solution of (MQVLI) and the problem is well-posed. The converse can be shown exactly as in Theorem 3.1. □

Comparing with Theorem 3.2 in Ref. [10], we see that besides treating a more general multivalued problem and imposing weaker conditions on the map S, we also avoid the monotonicity assumption.

We have analogous results for L-posedness.

Theorem 3.3 *Let the assumptions be as in Theorem 3.1, but with F η-pseudomonotone with respect to f, instead of η-monotone. Then,* (MQVLI) *is L-well-posed if and only if*

$$L_\varepsilon \ne \emptyset, \quad \forall \varepsilon > 0, \quad \text{and} \quad \lim_{\varepsilon \to 0} \operatorname{diam} L_\varepsilon = 0. \tag{6}$$

Lemma 3.3 *Let the assumptions be as in Theorem 3.3. Let $\{x_n\}_n \subseteq K$ be an L-approximating sequence. If $\{x_n\}_n$ converges to some $x_o \in K$, then x_o is a solution of* (MQVLI).

Proof Since $\{x_n\}_n$ is an L-approximating sequence, there exists a sequence $\{\varepsilon_n\}_n$ in $\mathbb{R}$ with $\varepsilon_n \downarrow 0$, such that

$$d(x_n, S(x_n)) \le \varepsilon_n \quad \text{and} \quad \langle v, \eta(x_n, y)\rangle + f(x_n) - f(y) \le \varepsilon_n,$$
$$\forall y \in S(x_n),\ v \in F(y),\ n \in N.$$

As in the proof of Lemma 3.1, $x_o \in S(x_o)$. For each $y \in \operatorname{int} S(x_o)$, one knows by virtue of Lemma 2.3 that $y \in S(x_n)$ for n sufficiently large. Thus, for all $v \in F(y)$,

$$\begin{aligned}&\langle v, \eta(x_o, y)\rangle + f(x_o) - f(y)\\ &\quad\le \liminf_n\{\langle v, \eta(x_n, y)\rangle + f(x_n) - f(y)\} \le \lim_n \varepsilon_n = 0.\end{aligned}$$

Applying Lemma 2.2 exactly as in Lemma 3.1, we infer that x_o is a solution of (MQVLI). □

Proof of Theorem 3.3 Assume that (6) holds. By Remark 2.2(a), every solution of (MQVLI) belongs to L_ε for all $\varepsilon > 0$. As in Theorem 3.1, this implies that the solution, if it exists, is unique. Also, L-approximating sequences exist, since for each sequence $\{\varepsilon_n\}_n$ in $\mathbb{R}$ with $\varepsilon_n \downarrow 0$ and each choice $x_n \in L_{\varepsilon_n}$, $\{x_n\}_n$ is an L-approximating sequence; thus it suffices to show that every L-approximating sequence for (MQVLI) converges to a solution of (MQVLI).

Let $\{x_n\}_n \subseteq K$ be an L-approximating sequence. As in the proof of Theorem 3.1, $\{x_n\}_n$ converges to some $x_o \in K$. By Lemma 3.3, x_o is a solution of (MQVLI) and the problem is L-well-posed.

Conversely, assume that the problem is L-well-posed; since F is η-pseudomonotone with respect to f, we know that the unique solution of (MQVLI) belongs to L_ε, thus $L_\varepsilon \neq \emptyset$ for each $\varepsilon > 0$. The proof concludes as in Theorem 3.1. □

Theorem 3.4 *Let the assumptions be as in Theorem 3.2. Assume further that F is η-pseudomonotone with respect to f. Then,* (MQVLI) *is L-well-posed if and only if condition* (6) *holds.*

Lemma 3.4 *Let the assumptions be as in Theorem 3.2. Let $\{x_n\}_n \subseteq K$ be an L-approximating sequence. If $\{x_n\}_n$ converges to some $x_o \in K$, then x_o is a solution of* (MQVLI).

Proof As in Lemma 3.3, we have that $x_o \in S(x_o)$ and

$$\langle v, \eta(x_n, y)\rangle + f(x_n) - f(y) \le \varepsilon_n, \quad \forall y \in S(x_n),\ v \in F(y),\ n \in \mathbb{N},$$

for some sequence $\{\varepsilon_n\}_n$ in $\mathbb{R}$ such that $\varepsilon_n \downarrow 0$.

Since $\liminf_n S(x_n) = S(x_o)$, for each $y \in S(x_o)$ there exists a sequence $y_n \in F(x_n)$, $n \in \mathbb{N}$, strongly converging to y. For each $n \in \mathbb{N}$ select $v_n \in F(y_n)$. Since F is (s, w^*) usc with weakly* compact values, we can find a bounded subnet

$\{v_\alpha\}_\alpha$ of $\{v_n\}_n$, weakly*-converging to some $v \in F(y)$. Hence $\langle v_\alpha, \eta(x_\alpha, y_\alpha)\rangle \to \langle v, \eta(x_o, y)\rangle$. Since by assumption

$$\langle v_n, \eta(x_n, y_n)\rangle + f(x_n) - f(y_n) \le \varepsilon_n, \quad n \in \mathbb{N},$$

we deduce from the above that

$$\forall y \in S(x_o),\ \exists v \in F(y) : \langle v, \eta(x_o, y)\rangle + f(x_o) - f(y) \le 0. \tag{7}$$

Note that the above relation is true for some (not all) $v \in F(y)$. We now apply Lemma 2.2 to $S_o = S_1 = S(x_o)$ and deduce that x_o is a solution of (MQVLI). □

Proof of Theorem 3.4 Assume that condition (6) holds. If $\{x_n\}_n \subseteq K$ is an L-approximating sequence, then as in the proof of Theorem 3.1, $\{x_n\}_n$ converges to some $x_o \in K$. Lemma 3.4 then implies that x_o is a solution of (MQVLI). Thus, the problem is well-posed. The converse can be proven as in Theorem 3.3. □

We now show that in the special case where E is finite-dimensional and K is compact, under suitable assumptions, (MQVLI) is L-well-posed if and only if it has a unique solution. It is known (Ref. [10]) that one cannot hope for such a result without assuming that K is compact.

Theorem 3.5 *Let $E = \mathbb{R}^k$, K be a convex compact subset of $\mathbb{R}^k$, let $f : K \to \mathbb{R}$ be convex and lsc, let $F : K \to 2^{\mathbb{R}^k}$ be upper hemicontinuous with nonempty, convex compact values, and let $\eta : K \times K \to E$ be continuous with $\eta(x,x) = 0$ for all $x \in K$ and such that $\langle v, \eta(x,\cdot)\rangle$ is concave for each $(v,x) \in F(K) \times K$. Let further $S : K \to 2^K$ be a continuous multifunction*[1] *such that, for each $x \in K$, $S(x)$ is nonempty, convex, and such that* $\dim(S(x)) \ge k-1$. *Then, the mixed quasivariational-like inequality* (MQVLI) *is L-well-posed if and only if it has a unique solution.*

Proof Assume that (MQVLI) has a unique solution z_o, and let $\{x_n\}_n$ be an L-approximating sequence. Since K is compact, this sequence has a subsequence converging to some x_o. If we show that x_0 is a solution, then we would have that $x_o = z_o$ and this would imply that the whole sequence converges to z_o.

So we can assume with no loss of generality that $x_n \to x_o$. Since $\{x_n\}_n$ is L-approximating, there exists a sequence $\{\varepsilon_n\}_n$ in $\mathbb{R}$ with $\varepsilon_n \downarrow 0$ and such that

$$\langle v, \eta(x_n, y)\rangle + f(x_n) - f(y) \le \varepsilon_n,$$
$$\forall y \in S(x_n),\ \forall v \in F(y),\ n \in \mathbb{N} \tag{8}$$

According to Proposition 2.1, for each $y \in \operatorname{ri} S(x_o)$ we can find a straight line passing through y and a sequence $\{y_n\}_n$ on this line, such that $y_n \to y$ and $y_n \in S(x_n)$, $n \in \mathbb{N}$: in fact, if $\dim S(x_o) = k-1$, it suffices according to Proposition 2.1 to consider

[1] In a finite-dimensional space, this is equivalent to our usual assumption that for $x_n \to x_o$, $S(x_n)$ Mosco converges to $S(x_o)$. In view of the compactness of K, this is also equivalent to the assumption that S is closed and lsc, used in Ref. [10].

any straight line that does not belong to the affine subspace generated by $S(x_o)$; if $\dim S(x_o) = k$, then according to Lemma 2.3 we can take any line through x_o, and $y_n = y$ for all $n \in \mathbb{N}$. Choose $v_n \in F(y_n)$, $n \in \mathbb{N}$. Since F is upper hemicontinuous, by choosing a subsequence if necessary, we may assume with no loss of generality that $\{v_n\}_n$ converges to some $v \in F(y)$. From (8), it follows that

$$\langle v_n, \eta(x_n, y_n)\rangle + f(x_n) - f(y_n) \leq \varepsilon_n, \quad n \in \mathbb{N}.$$

According to our assumptions, $f(x_o) \leq \liminf f(x_n)$, $\lim f(y_n) = f(y)$ (because f as a convex and lsc function, is continuous on any straight line), $\lim\langle v_n, \eta(x_n, y_n)\rangle = \langle v, \eta(x_o, y)\rangle$ and $\lim \varepsilon_n = 0$. Hence,

$$\forall y \in \operatorname{ri} S(x_o),\ \exists v \in F(y) : \langle v, \eta(x_o, y)\rangle + f(x_o) - f(y) \leq 0.$$

Using now Lemma 2.2 for $S_o = S(x_o)$ and $S_1 = \operatorname{ri} S(x_o)$, we infer that x_o is a solution of (MQVLI), as desired. □

Corollary 3.1 *Let the assumptions be as in Theorem* 3.5. *Assume further that F is η-monotone. Then*, (MQVLI) *is well-posed if and only if it has a unique solution.*

Proof Assume that (MQVLI) has a unique solution x_o. Let $\{x_n\}_n$ be an approximating sequence for (MQVLI). Then the η-monotonicity of F immediately implies that $\{x_n\}_n$ is also L-approximating. By Theorem 3.5, (MQVLI) is L-well-posed; hence, $\{x_n\}_n$ converges to x_o. Thus, (MQVLI) is well-posed. □

4 Well-Posedness in the Generalized Sense

The concept of well-posedness has often to be amended to accommodate the possible existence of more than one solutions of the problem under investigation (optimization problem, variational inequality or more general). Generalizing the definitions in Refs. [6] and [10], we give the following definition.

Definition 4.1 A mixed quasivariational-like inequality is called well-posed (resp. L-well-posed) in the generalized sense if the set of solutions Ω of (MQVLI) is nonempty and every approximating sequence (resp., L-approximating sequence) $\{x_n\}_n$ has a subsequence strongly converging to a solution of (MQVLI).

For the study of well-posedness in the generalized sense one usually makes use of the Kuratowski measure of noncompactness:

Definition 4.2 Given a subset A of a metric space $(X.d)$, the Kuratowski measure of noncompactness $\alpha(A)$ is the infimum of $\varepsilon > 0$ such that there exists a finite covering of A by sets of diameter at most ε.

With the help of this measure of noncompactness, we can give necessary and sufficient conditions for well-posedness in the generalized sense, under continuity and generalized monotonicity conditions which are the same as the ones we used for well-posedness.

Theorem 4.1 *Let the assumptions be as in Theorems 3.1 or 3.2. Then*, (MQVLI) *is well-posed in the generalized sense if and only if*

$$Q_\varepsilon \neq \emptyset, \quad \forall \varepsilon > 0 \quad \textit{and} \quad \lim_{\varepsilon \to 0} \alpha(Q_\varepsilon) = 0. \tag{9}$$

Proof Assume that (MQVLI) is well-posed in the generalized sense. Then $Q_\varepsilon \supseteq \Omega \neq \emptyset$ for all $\varepsilon > 0$. Assume that $\lim_{\varepsilon \to 0} \alpha(Q_\varepsilon) = 0$ does not hold; then since $\varepsilon \geq \varepsilon'$ implies $Q_\varepsilon \supseteq Q_{\varepsilon'}$ and $\alpha(Q_\varepsilon) \geq \alpha(Q_{\varepsilon'})$, there exists $\delta > 0$ such that $\alpha(Q_\varepsilon) > \delta$ for all $\varepsilon > 0$. Define a sequence $\{x_n\}_n$ inductively as follows: choose $x_1 \in Q_1$. Having chosen $x_1, \ldots, x_{n-1}$, we know that the spheres $B(x_i, \delta/2)$, $i = 1, \ldots, n-1$, do not cover $Q_{\frac{1}{n}}$, since $\alpha(Q_{\frac{1}{n}}) > \delta$ and $\operatorname{diam} B(x_i, \delta/2) = \delta$; then, choose

$$x_n \in Q_{\frac{1}{n}} \Big\backslash \bigcup_{i=1}^{n-1} B(x_i, \delta/2).$$

By construction, $\|x_n - x_m\| \geq \delta/2$ for all $n, m \in \mathbb{N}$, thus $\{x_n\}_n$ has no converging subsequence. But $x_n \in Q_{\frac{1}{n}}$ hence $\{x_n\}_n$ is an approximating sequence for (MQVLI), a contradiction.

Conversely, assume that (9) holds. Let $\{x_n\}_n$ be an approximating sequence for (MQVLI). Then there exist $\varepsilon_n > 0$ such that $x_n \in Q_{\varepsilon_n} \subseteq \overline{Q_{\varepsilon_n}}$, $n \in \mathbb{N}$. Since $\lim_n \alpha(\overline{Q_{\varepsilon_n}}) = \lim_n \alpha(Q_{\varepsilon_n}) = 0$, this implies that $\{x_n\}_n$ has a subsequence converging to some $x_0 \in K$ (cf. Ref. [3, p. 4]). Using Lemmas 3.1 or 3.2, depending on our set of assumptions, we deduce that x_o is a solution and (MQVLI) is well-posed in the generalized sense. □

Remark 4.1 In the general theory of measures of noncompactness it is shown that any measure of noncompactness μ has the following property: any sequence $\{x_n\}_n$ with the property $\lim_n \mu(\{x_n, x_{n+1}, \ldots\}) = 0$ has a cluster point (Ref. [3, p. 11]). It is clear from the above proof that, if

$$Q_\varepsilon \neq \emptyset, \quad \forall \varepsilon > 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \mu(Q_\varepsilon) = 0 \tag{10}$$

holds with any measure of noncompactness μ, then (MQVLI) is well-posed in the generalized sense. Indeed, if $\{x_n\}_n$ is an approximating sequence then $\{x_n, x_{n+1}, \ldots\} \subseteq Q_{\varepsilon_n}$ hence $\lim_n \mu(\{x_n, x_{n+1}, \ldots\}) = 0$ holds, so $\{x_n\}_n$ has a converging subsequence; then the last part of the proof of Theorem 4.1 shows that this subsequence converges to a solution of (MQVLI). However, the converse is not true for some measures of noncompactness; if (MQVLI) is well-posed in the generalized sense but ill-posed and the assumptions of Theorem 3.1 are satisfied, then (10) does not hold for the measure of noncompactness $\mu(A) = \operatorname{diam}(A)$. However, if μ is a regular measure of noncompactness, then there exists a constant $c > 0$ such that for each nonempty bounded set A, $\mu(A) \leq c\alpha(A)$ (cf. pp. 7 and 12 in Ref. [3]). In this case, condition (9) implies condition (10). We conclude that Theorem 4.1 holds if α is replaced by any regular measure of noncompactness μ.

An analogous result holds for L-well-posedness:

Theorem 4.2 *Let the assumptions be as in Theorems 3.3 or 3.4. Then*, (MQVLI) *is L-well-posed in the generalized sense if and only if*

$$L_\varepsilon \neq \emptyset, \quad \forall \varepsilon > 0 \quad \textit{and} \quad \lim_{\varepsilon \to 0} \alpha(L_\varepsilon) = 0. \tag{11}$$

Proof Assume that (MQVLI) is L-well-posed in the generalized sense. Since F is η-pseudomonotone with respect to f, one has $L_\varepsilon \neq \emptyset$, $\forall \varepsilon > 0$. Exactly as in Theorem 4.1, if we assume that $\lim_{\varepsilon \to 0} \alpha(L_\varepsilon) > 0$ we deduce that there exists an L-approximating sequence which has no convergent subsequence. Thus (11) holds.

Conversely, assume that (11) holds. As in the proof of Theorem 4.1, we deduce that every L-approximating sequence has a convergent subsequence. In virtue of Lemmas 3.3 and 3.4, this limit is a solution of (MQVLI), thus the problem is L-well-posed in the generalized sense. □

Finally, we show that L-well-posedness in the generalized sense can be automatically valid in finite dimensions.

Theorem 4.3 *Let $E = \mathbb{R}^k$, K be a convex compact subset of $\mathbb{R}^k$, let $f : K \to \mathbb{R}$ be convex and continuous, let $F : K \to 2^{\mathbb{R}^k}$ be upper semicontinuous with nonempty, convex compact values, and let $\eta : K \times K \to \mathbb{R}^n$ be continuous with $\eta(x,x) = 0$ for all $x \in K$ and such that $\langle v, \eta(x,\cdot)\rangle$ is concave for each $(v,x) \in F(K) \times K$. Let further $S : K \to 2^K$ be a continuous multifunction such that, for each $x \in K$, $S(x)$ is nonempty and convex. Then, the mixed quasivariational-like inequality* (MQVLI) *is well-posed and L-well-posed in the generalized sense.*

Proof We first show that Ω is nonempty. The proof of this fact actually follows from the proof of Theorem 3.1 in Ref. [4] with small adjustments and is reproduced for convenience of the reader. Since F is upper semicontinuous with compact values, the set $F(K)$ is compact, hence also the set $D = \operatorname{co} F(K)$ is compact. It is easy to check that $\langle v, \eta(x,\cdot)\rangle$ is concave for each $(v,x) \in D \times K$. Define the function $\varphi : K \times K \times D \to \mathbb{R}$ by $\varphi(y,x,v) = \langle v, \eta(x,y)\rangle - f(y)$. Then, φ is continuous, hence by the maximum theorem the map $T : K \times D \to 2^K \setminus \{\emptyset\}$ defined by

$$T(x,v) = \operatorname*{argmax}_{y \in S(x)} \varphi(y,x,v)$$

is usc with nonempty compact values. Since $\varphi(\cdot,x,v)$ is concave, $T(x,v)$ is a convex set for $(x,v) \in K \times D$. Define a map $T_1 : K \times D \to 2^{K \times D}$ by

$$T_1(x,v) = \{(y,w) : y \in T(x,v), w \in F(x)\}.$$

Then, T_1 is usc with nonempty compact convex values. By the Kakutani fixed-point theorem, there exists $(x_o, v_o) \in K \times D$ such that $(x_o, v_o) \in T_1(x_o, v_o)$. This implies that $v_o \in F(x_o)$ and $x_o \in T(x_o, v_o)$, i.e.,

$$\varphi(y, x_o, v_o) \le \varphi(x_o, x_o, v_o) = -f(x_o), \quad \forall y \in S(x_o).$$

Thus, $x_o \in \Omega$ so $\Omega \neq \emptyset$.

Now let $\{x_n\}_n$ be an approximating sequence. Since K is compact, there exists a subsequence converging to some limit x_1. By Lemma 3.2, $x_1 \in \Omega$. Hence the problem is well-posed in the generalized sense. Likewise, if $\{x_n\}_n$ is an L-approximating sequence, then it has a converging subsequence which by Lemma 3.4 converges to an element of Ω. Hence the problem is L-well-posed in the generalized sense. □

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